Final Report

Junior Trimester Program
Kinetic Theory

This is the final report for the group *LFD and Inelastic Boltzmann equation* at the Junior Trimester Program (JTP) at the Hausdorff Institute for Mathematics (HIM) in the spring of 2019. The members of our group were Ricardo J. Alonso (Jun. 30 – Jul. 17), Véronique Bagland (Jul. 01 – Jul. 13) and Bertrand Lods (Jul. 01 – Jul. 06).

Scientific activities

The group acknowledges the excellent working conditions provided by the Hausdorff Research Institute for Mathematics (HIM). We are particularly grateful for the funding provided for the stay and the facilities offered by HIM. We feel very privileged to have had this opportunity to work, even for a short period, at the HIM. The whole JTP has been a very productive, and pleasant experience and we enjoyed being in touch during our stay with the other group members ~~during our stay~~. During our period of stay, no workshop were organised at the HIM but we attended various of the *Trimester Seminars*. A member of the group (B. Lods) presented some of the previous results of the group in a talk given within the *Seminar Series Coagulation-fragmentation*.

Research and output

During our stay at the HIM we discussed and worked on the following questions related to the kinetic description of *quantum plasmas and gases* as well as *granular gases*:

(1) ***Long-time behavior of Landau-Fermi-Dirac equation.*** As planned in the research project, we have been able to make progress on the study of Landau-Fermi-Dirac equation. Such a model is a modification of the Landau equation for collisional plasma to quantum particles taking into account the Pauli exclusion principle. The three group members finalise at the HIM a first work regarding the long-time behaviour of the solution to the spatially homogeneous equation for interactions corresponding to the so-called *hard potentials*. The results obtained in this paper has been published in a high-quality international journal [1]. We exhibit in this work an exponential convergence towards to equilibrium by exploiting the appearance and uniform boundedness of moments and regularity norms for the solutions to Landau-Fermi-Dirac equation. The rate of convergence is obtained by combining a careful spectral analysis (for close-to-equilibrium solutions) with a thorough analysis of the entropy production properties of the collision operator. Such an approach is robust enough to be applied successfully to the more delicate case of *soft potentials* interactions for which the convergence rate is not expected to be exponential anymore. We started, during our stay at HIM, to work on a general study of the Landau-Fermi-Dirac equation for soft potential interactions. A few months after the JTP, we had the opportunity

to get the help of Laurent Desvillettes on this research topics and the collaboration results in one submitted paper [2] and a paper which is at its final stage [3] before submission. In the first of this paper, we give a complete and unified study of the entropy production for the Landau-Fermi-Dirac equation (with hard or moderately soft potentials) in terms of a weighted relative Fisher information adapted to this equation. Such estimates are used for studying the large time behaviour of the equation, as well as for providing new a priori estimates (in the soft potential case). In the second paper, we establish a nearly optimal algebraic relaxation to equilibrium for solutions to the LFD equation with moderately soft potentials. The analysis is highly non trivial and combines uniform in time estimates for statistical moments, L^p-norm generation and Sobolev regularity together with an innovative level set analysis in the spirit of De Giorgi.

(2) ***Entropy production for Quantum Boltzmann equation.*** In the spirit of the previous point, we also started during our stay at HIM, a comprehensive analysis of the entropy production for the Quantum Boltzmann equation, for both Fermi-Dirac and Bose-Einstein description. The geometry of the collision here makes the analysis significantly more technical than the one of the related Landau equation. We obtained already encouraging results at the HIM but such partial results still need to be finalised and complemented with a careful study of the solutions to the Quantum Boltzmann equation. Indeed, several of the entropy production estimates we obtained so far are strongly related to some *a priori* regularity estimates. To apply the newly obtained tools to the study of the long-time behaviour of the associated Quantum Boltzmann equation, we need to prove that solutions to such equations do indeed satisfy those regularity estimates. This is a quite delicate problem which is, for the members of the group, a challenging research project on a mid-term basis.

(3) ***Regularity theory and uniqueness of steady states for 1D inelastic Boltzmann.*** In collaboration with José A. Cañizo and Sebastian Throm (member of the group lead by J. A. Cañizo) we started to work at HIM on the propagation of regularity and uniqueness of the steady state for a one dimensional Boltzmann equation with dissipative interactions on which our group worked in recent years. These two questions are well-understood for Boltzmann equation with dissipative interactions in higher dimension, but their analysis for the one-dimensional version of the equation is quite delicate. For the propagation of regularity, the main obstacle is the lack of particle scattering. Indeed, mathematically speaking, particle scattering represent some kind of diffusion process that help to find coercivity estimates. Contrary to particle systems in two or more dimensions, this is not the case in 1D inelastic Boltzmann where such coercivity is very weak. As far as the uniqueness of the steady state is concerned, the question is answered in higher dimension by some perturbative argument on the inelasticity parameter and exploiting the results known for the classical Boltzmann equation (with conservative interactions). Such an approach does not apply to the 1D inelastic Boltzmann equation since its elastic counterpart

is meaningless. We discussed in HIM the possibility of tackling both these questions using perturbative techniques not related to the inelasticity parameter but with the strength of the potential interactions. Indeed, in the case of constant interaction potentials (pseudo-Maxwellian interactions), the uniqueness and propagation of regularity are known to hold thanks to a clever use of Fourier analysis and Fourier-based distances. We believe it should be possible to exploit these known results as a pivot for a perturbative approach when dealing with hard potential interactions whose strength γ is small enough (the pseudo-Maxwellian case corresponding to $\gamma = 0$). This approach is completely new for the study of the Boltzmann equation even it is reminiscent to similar questions addressed in the study of coagulation-fragmentation processes. Before attacking this perturbative approach, we started to strengthen the results obtained in the pseudo-maxwellian and reformulate them in some more robust Banach space framework (not depending anymore on Fourier-based metrics which are ineffective for hard potentials). The results obtained so far are already satisfactory but we aim to complement them with the aforementioned implementation of the perturbative approach which is much more delicate. This project is still an active field of research for our group even if the progresses made in the previous two projects delayed us a bit in this specific one.

(4) ***Contractivity for Smoluchowski's coagulation equation*** Inspired by the discussions on the Fourier-based metrics for the 1D-Boltzmann equation and the analogy with models for coagulation-fragmentation, we also discussed at the HIM the adaptation of these metrics to the study of Smoluchowski's equation. These discussions result in the publication of the paper [4] which revisit the study of the so-called solvable kernels for Smoluchowski's equation introducing for this purpose metrics based upon the Laplace transform technique instead of Fourier transform. Together with José A. Cañizo and Sebastian Throm, research is still active in this direction, in particular trying to adapt the techniques introduced in [4] to some other kernels for which uniqueness of the steady state is known to hold (inverse power law kernel).

References

[1] R. J. Alonso, V. Bagland, & B. Lods, Long time dynamics for the Landau-Fermi-Dirac equation with hard potentials, *J. Differential Equations* **270** (2021), 596–663.

[2] R. J. Alonso, V. Bagland, L. Desvillettes, & B. Lods, About the use of Entropy dissipation for the Landau-Fermi-Dirac equation, submitted for publication, 2020.

[3] R. J. Alonso, V. Bagland, L. Desvillettes, & B. Lods, About the Landau-Fermi-Dirac equation with moderately soft potentials, work in progress, 2021.

[4] J. Canizo, B. Lods, & S. Throm, Contractivity for Smoluchowski's coagulation equation with solvable kernels, *Bulletin of the London Mathematical Society*, to appear, 2020.