Report on the Project *Confined Kinetic Equations*

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1 Scientific goals

Kinetic theory allows the description of a gas (or any physical, biological or abstract substance with similar properties) at a mesoscopic scale, a scale which is intermediate between microscopic and macroscopic. The impacts range from very fundamental questions to practical applications: theoretical bridge between the microscopic and macroscopic theories (reunification of Physics), compromise of intermediate mathematical and numerical complexity, modelling of situations where the fluid approximation is too rough (such as rarefied gas), derivation of new fluid-like models in various contexts (Biology, Social sciences, ...), among others.

In practice, in the vast majority of applications, the gas considered is in interaction with a solid object: sometimes the object of interest is actually the solid itself (eg. in spatial engineering), in other contexts the gas is contained in a box and interacts with the walls. It is therefore essential to study the problem with boundary conditions.

From the mathematical point of view, the situation in bounded domains is extremely complex. In typical cases, the presence of a boundary creates a singularity. The singularity is formed at the boundary, and propagates along the trajectories: therefore, the geometry of the domain plays a fundamental role [5, 6]. In the case of a collisional model, these singularities may be somehow compensated by the regularizing effect of the collisional operator and the solution might be more regular inside the domain than at the boundary. Conversely, if the collisions are neglected, one expects a more singular solution.

The goal of this project is to quantify precisely this singularity by developing a maximal regularity theory with the help of adequate mathematical tools: regularity functional spaces (Sobolev, Hölder, ...) and adapted localisation of the singularity (*kinetic distance*, geometrical description of singular trajectories, etc.).

We started with the most basic problem we could think of: a collisionless equation. Consider then the free transport equation in a bounded domain Ω,

$$\partial_t F + v \cdot \nabla_x F = 0,$$

where $F = F(t, x, v)$ is the density (in space $x \in \Omega \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$) of particles at time $t \geq 0$. This equation is supplemented with the specular boundary conditions (billiard-like reflections),

$$F(t, x, v) = F(t, x, \mathrm{R}_x v), \qquad \text{for } x \in \partial\Omega,$$

where $\mathrm{R}_x = \mathrm{I}_3 - 2\nu(x) \otimes \nu(x)$ is the reflection through the tangential plane to the boundary at the point $x \in \partial\Omega$ (and $\nu(x)$ is the outward normal at point x).

As simple as this equation might look, several fundamental questions are still unanswered such as the maximal regularity of solutions − even in elementary convex domains − due to the conservative nature of the boundary condition [1] as opposed to contracting boundary operators.

We decided to tackle the question of the maximal regularity in two simple geometries to start with: the case of a flat domain (half-space) and the case of the unit ball. Indeed, see [3, 4], the symmetries of the ball sensibly simplify the analysis − for instance due to the fact that the trajectories of the free transport in this setting are 2D − without depriving the problem of its key difficulties. We expected, through the unit ball, to identify the fundamental phenomena which impact the Hölder/Sobolev regularity of solutions. This would allow to tackle the general problem with a much better understanding of the underlying mechanics.

2 Advances

We developed a geometrical approach, based on the unfolding of characteristics, that allows to treat the regularity up to the boundary, grazing boundary excluded, in any convex domain. We complemented this method by an operator algebra approach to treat the singularities at the grazing boundary in flat domains (half-space). Thanks to these methods, we obtained a **maximal regularity theory of the free transport problem in flat domains, in terms of Hölder and (local) Sobolev spaces** [2].

To go beyond flat geometries, we considered the unit ball. We first obtained the regularity far from the grazing (thanks to the unfolding method). We then analyzed that the case of the ball contains two main difficulties: (1) the infinite number of reflections in finite time and (2) the geometry. To understand former we introduced a toy problem that contains difficulty (1) but not difficulty (2): we consider the free transport in a flat domain, and added a force field that attracts particles towards the boundary, so that each particle is attracted to the boundary, then reflected away, attracted to, reflected away, and so on. As for the case without force, we obtained a **maximal regularity theory of the transport problem with attractive force in flat domains, in terms of Hölder and (local) Sobolev spaces** [2].

3 Long-term goals and other projects

The main continuation of this project is to consider the case of the unit ball. As explained before, the main difficulty of the ball lies in the geometrical aspects. In the longer term, we will consider more general geometries, and, finally, add a collision operator (eg. Boltzmann operator). The natural follow-up is then to consider more realistic boundary conditions such as the Maxwell boundary conditions, in which the temperature of the boundary influences the gas.

A. T. would like to also mention an ongoing work with M. BREDEN and M. HERDA that was initiated thanks to fortuitous discussions during the program. The purpose of this work is a link between kinetic theory and cross-diffusion systems that brings new understanding in the modelling.

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