Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

III. Positivity in Temperley-Lieb algebras and dual Garside structures on Artin-Tits groups

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Junior Hausdorff Trimester
"Symplectic Geometry and Representation Theory"
Bonn, October 2017

Topological models in the classical types

Temperley-Lieb algebras

Garside monoids and groups

Dual braid monoids

Positivity in Temperley-Lieb algebra

Topological descriptions of Mikado braids

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- The topological definition of Mikado braids is, as we will see today, useful and even necessary in some cases to show results involving Mikado braids.

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- The topological definition of Mikado braids is, as we will see today, useful and even necessary in some cases to show results involving Mikado braids.
- **Question:** Is there a topological characterization of Mikado braids in the above mentioned cases ?

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- If W is a Coxeter group of type B_n, then there are (at least) two realizations of $B(W)$ by Artin-like braids.

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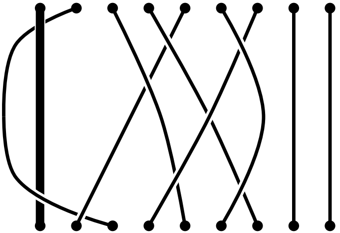
First model: Artin braids on $n+1$ strands with an unbraided first strand.



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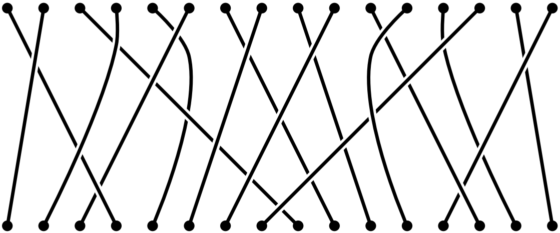
Second model: symmetric braids on $2n$ strands



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Theorem (Digne-G., 2015)

Let $\beta \in B(W^\Gamma)$. The following are equivalent

1. *The braid β is a Mikado braid.*
2. *There is an Artin braid in $B(W)$ representing β, such that one can inductively remove pairs of symmetric strands, one of the two strands being above all the other strands (so that the symmetric one is under all the other strands).*

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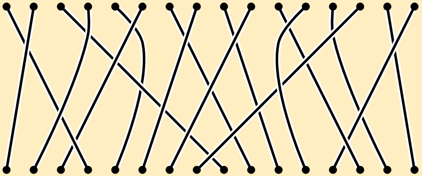
Second model: symmetric braids on 2n strands



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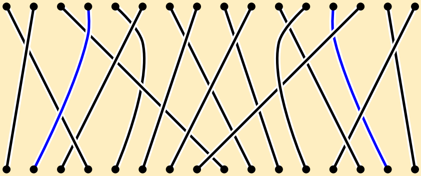

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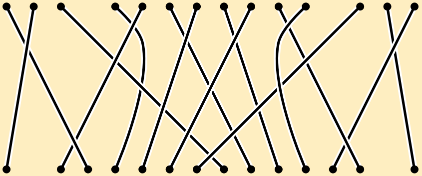

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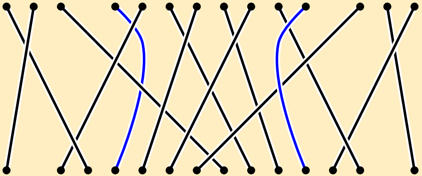

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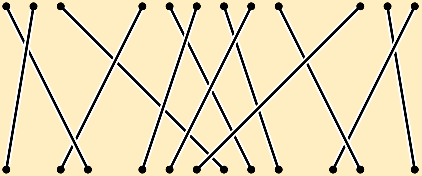

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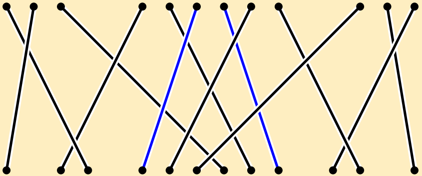

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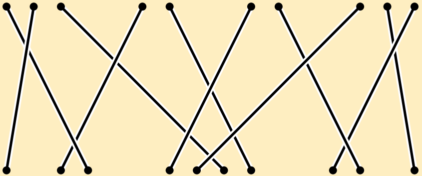

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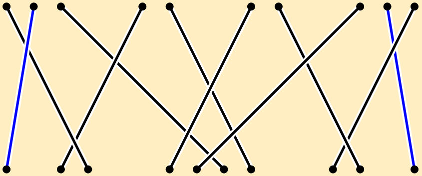

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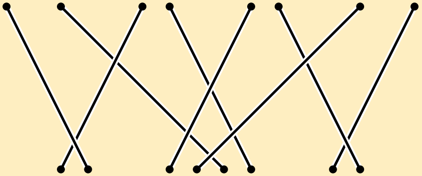

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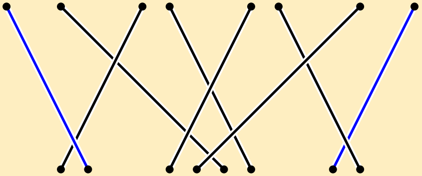

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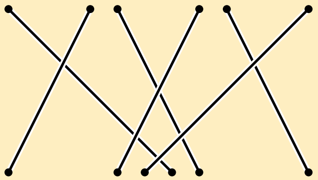

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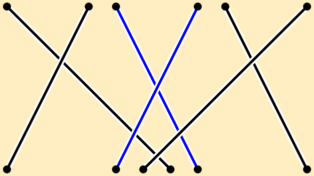

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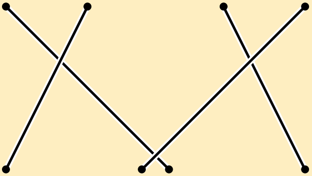

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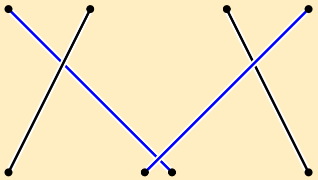

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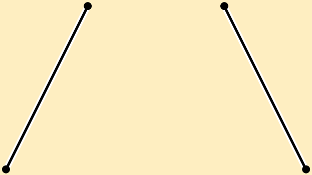

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Let $\beta \in B(W') \subseteq \overline{B}$. The following are equivalent.

1. *The braid β is Mikado.*
2. *There is a Mikado braid $\beta' \in B(W^\Gamma)$ such that $\beta = \pi(\beta')$, where $\pi : B(W^\Gamma) \twoheadrightarrow \overline{B}$ is the quotient map.*

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- It can be realized as a quotient of $\mathcal{H}(\mathfrak{S}_n)$ in at least two different ways.

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- It can be realized as a quotient of $\mathcal{H}(\mathfrak{S}_n)$ in at least two different ways. It is either the quotient by the two-sided ideal I generated by the C_{sts}, for all $s, t \in S$ such that $st \neq ts$, or by the two-sided ideal I' generated by the C'_{sts}, for all $s, t \in S$ such that $st \neq ts$.

Temperley-Lieb algebras

- Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The *Temperley-Lieb algebra* TL_n is the associative, unital $\mathcal{A}$-algebra with generators $b_1, b_2, \ldots, b_{n-1}$ and relations
  1. $b_i b_{i\pm 1} b_i = b_i$, $\forall i$,
  2. $b_i b_j = b_j b_i$, $\forall i, j$ such that $|i-j| > 1$,
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- We will work with the quotient map $\theta : \mathcal{H}(\mathfrak{S}_n) \to \mathrm{TL}_n$ corresponding to the ideal I.

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- Set $\delta := v + v^{-1}$.

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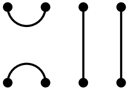
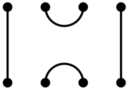
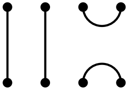
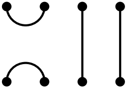
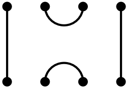
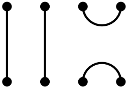
Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

- Multiplication = concatenation

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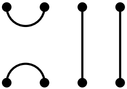
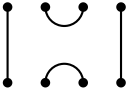
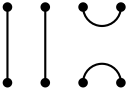
$b_1 =$  , $b_2 =$  , $b_3 =$ 

- Multiplication = concatenation
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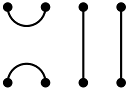
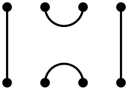
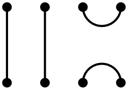
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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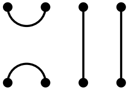
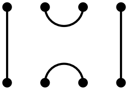
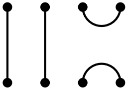
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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$$b_1 b_2 b_1 = b_1$$

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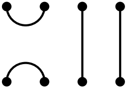
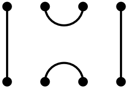
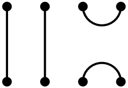
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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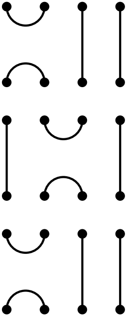
b_1

b_2

b_1



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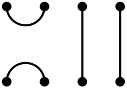
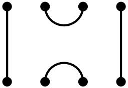
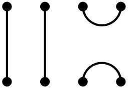
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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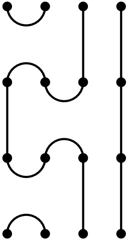

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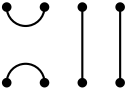
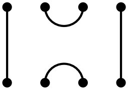
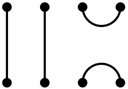
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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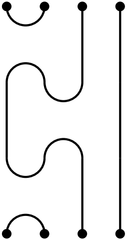

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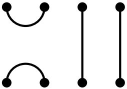
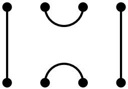
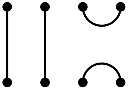
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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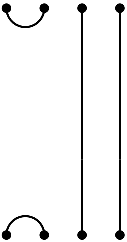

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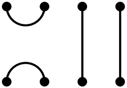
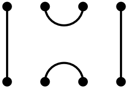
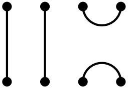
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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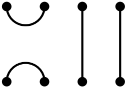

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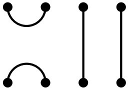
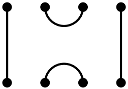
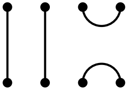
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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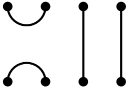
b_1 

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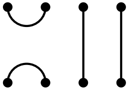
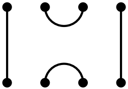
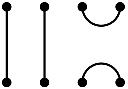
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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$$b_3 b_1 = b_1 b_3$$

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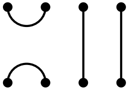
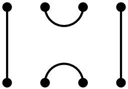
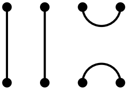
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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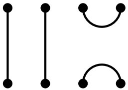
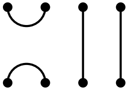
b_3 

b_1 

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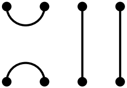
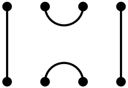
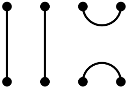
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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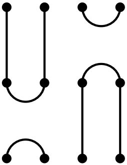

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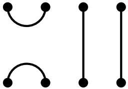
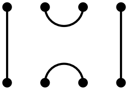
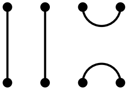
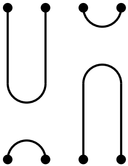
$b_1 =$ , $b_2 =$ , $b_3 =$ 

- Multiplication $=$ concatenation
- Multiplication by δ $=$ add a circle in the diagram



$$b_3 b_1 = b_1 b_3$$

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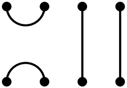
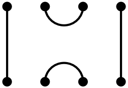
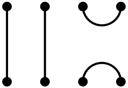
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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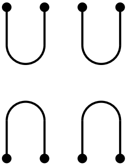

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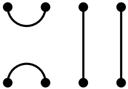
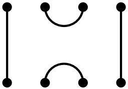
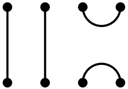
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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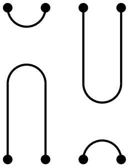

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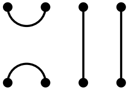
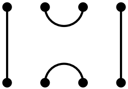
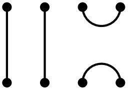
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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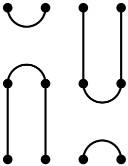

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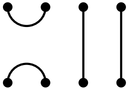
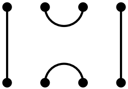
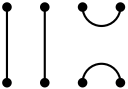
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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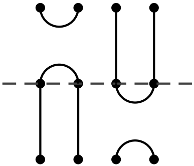

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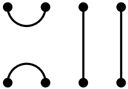
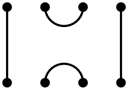
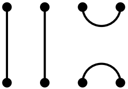
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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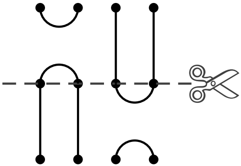

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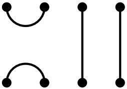
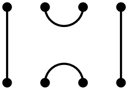
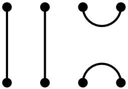
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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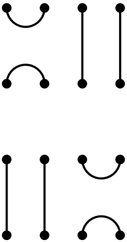

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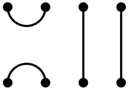
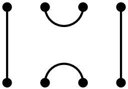
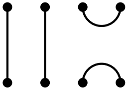
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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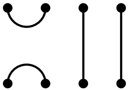
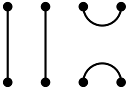
b_1 

b_3 

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Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

III. Positivity in Temperley-Lieb algebras and dual Garside structures on Artin-Tits groups

Thomas Gobet

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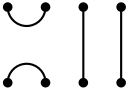
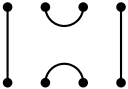
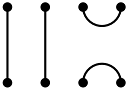
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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$$b_1^2 = \delta b_1$$

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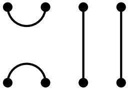
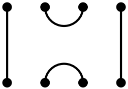
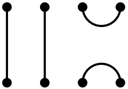
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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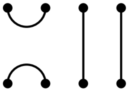
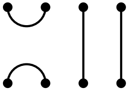
b_1 

b_1 

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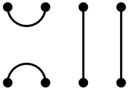
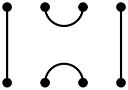
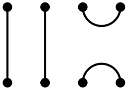
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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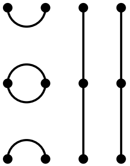

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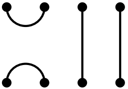
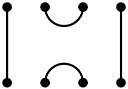
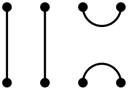
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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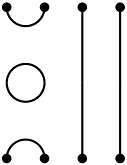

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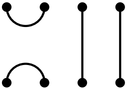
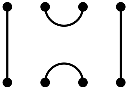
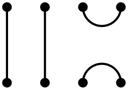
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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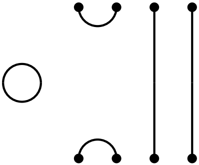

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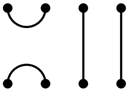
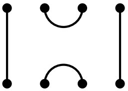
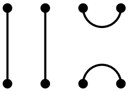
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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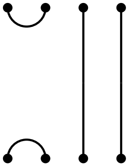
δ 

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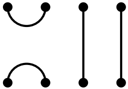
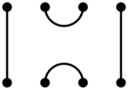
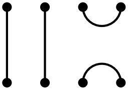
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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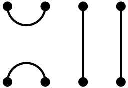
δ 

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Multiplying generators b_i yields (linear combinations of) various diagrams. In case $n = 4$, there are 14 possible diagrams:

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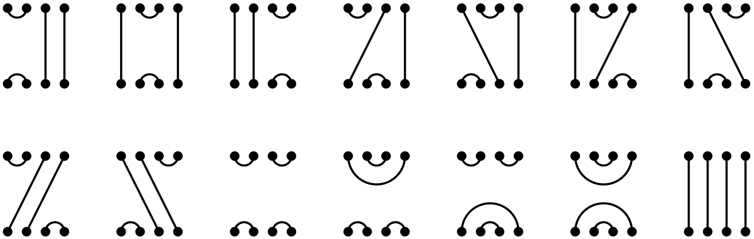

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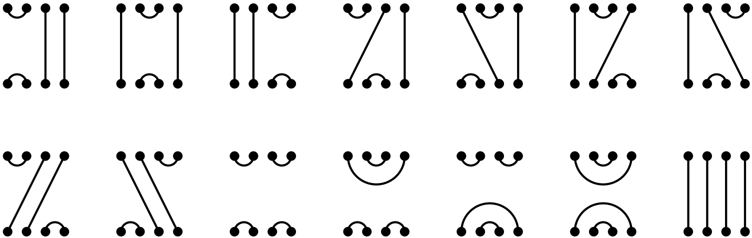

These diagrams, which form a basis of the obtained diagram algebra, form a basis of the algebra.

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Let $x \in \mathfrak{S}_n$. We say that x is *fully commutative* if one can pass from any reduced expression of x to any other just by applying a sequence of commutation relations $s_i s_j = s_j s_i$, $|i-j| > 1$.

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- Let x be fully commutative, let $s_{i_1} s_{i_2} \cdots s_{i_i}$ be a reduced expression. As a consequence of the TL defining relations, the element

$$b_x := b_{i_1} b_{i_2} \cdots b_{i_k}$$

is well-defined.

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The basis $\{b_x\}_{x\in \mathrm{FC}(\mathfrak{S}_n)}$ *is (up to signature) the projection of the basis* $\{C_w\}_{w\in W}$ *of* $\mathcal{H}(\mathfrak{S}_n)$ *under* θ*,*

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A monoid M is *left-cancellable* (resp. *right-cancellable*) if whenever $ab = ac$ (resp. $ba = ca$) with $a, b, c \in M$, we have $b = c$.

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Let M be a monoid, $a \in M$. We say that $b \in M$ *left-divides* (resp. *right-divides*) $a \in M$ if there is $c \in M$ such that $bc = a$ (resp. $a = cb$).

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Definition (Noetherian divisibility)

We say that the divisibility in a monoid M is *Noetherian* if there exists $\lambda : M \longrightarrow \mathbb{Z}_{\geq 0}$ such that $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $g \neq 1 \Rightarrow \lambda(g) \neq 0$.

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- Note that in a monoid M with Noetherian divisibility, every nontrivial element has infinite order and there are no nontrivial invertible elements.

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- Under the above assumptions, one can define a group $G(M)$ of left-fractions of M, that is, whose elements are $f^{-1}g$ for $f, g \in M$, in which M embeds.

Spherical Artin-Tits groups are Garside groups

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Every finite Coxeter group has a unique element w_0 of maximal length.

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Theorem (Garside, 1967)

The Artin-Tits group $B(W)$ is a Garside group, with corresponding Garside monoid $(B(W)^+, \Delta)$ and we have $\{\text{Div. of } \Delta\} = \{\mathbf{x}, x \in W\}$.

Example: $W = \mathfrak{S}_3$.

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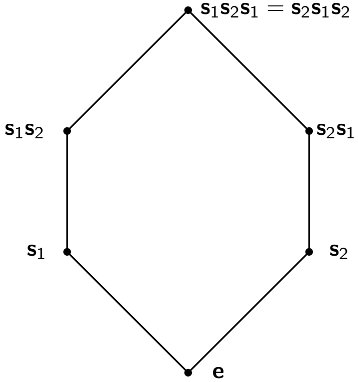

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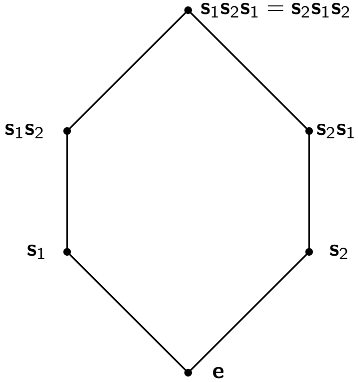

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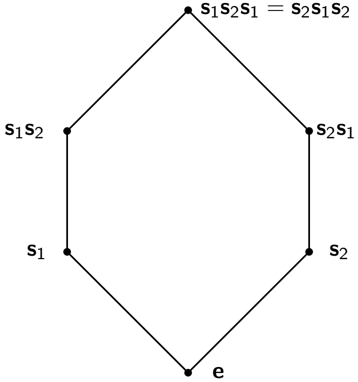

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Alternative Garside structures

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- In Garside groups, one can show the existence of normal forms for the elements of the group.

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An alternative presentation of the positive braid monoid

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An alternative presentation of the positive braid monoid

- Define the *left weak order* $\leq_S$ on W by setting $x \leq_S y$ if $\ell(x) + \ell(x^{-1}y) = \ell(y)$.

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- The idea of dual braid monoids is to replace the generating set S of W by the set T of all the reflections, and build a monoid as above.

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Definition (Coxeter element)

A *Coxeter element* is a product of all the elements of S, in some order. Is is an n-cycle.

Absolute order on a Coxeter group

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Proposition (Biane, 1997)

The poset $(P_c, \leq_T)$ is a lattice, isomorphic to the noncrossing partition lattice.

Noncrossing partitions

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Definition

A *noncrossing partition* is a partition π of $\{1, 2, \ldots, n\}$ such that the following never happens: B_1, B_2 are two distinct blocks of π with $i, j \in B_1$, $k, \ell \in B_2$ and $i < k < j < \ell$.

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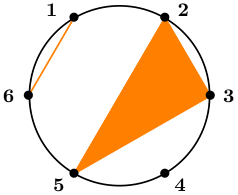

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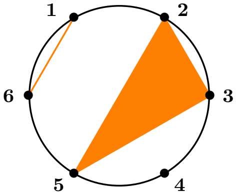

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- Ordering each polygon in clockwise order yields a permutation $\sigma(\pi)$ in $\mathfrak{S}_n$ (polygons correspond to cycles).

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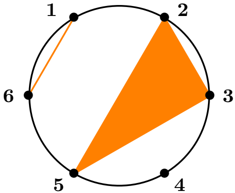

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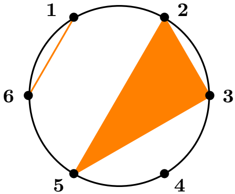

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- We have $x \leq_T c = (1, 2, \ldots, n)$ if and only if there is a noncrossing partition π with $x = \sigma(\pi)$. The order $\leq_T$ corresponds to the refinement order on noncrossing partitions.

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- Since all Coxeter elements are conjugate and T is stable by conjugation, $(P_c, \leq_T)$ is isomorphic to the n.p. lattice for all c.

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- Everything which we did can be done for every finite Coxeter group: replace the transpositions by the reflections to define ℓ_T and $\leq_T$, Coxeter elements are products of the elements of S.

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Definition (dual braid monoids)

The *dual braid monoid* B_c^* is defined by

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- **Question**: is there a nice formula for $\iota(x_c)$ in the generators $\mathbf{S}$?

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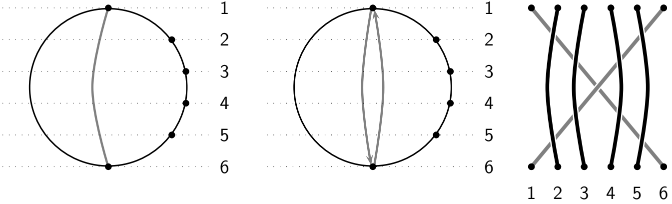

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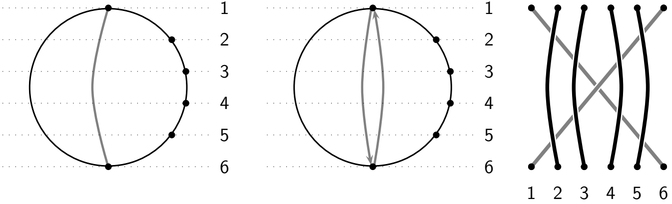

- For every $x \in P_c$, the Artin braid x_c is obtained first by taking a T-reduced expression $x = t_1 t_2 \cdots t_k$, $t_i \in T$ and then we have $x_c = (t_1)_c (t_2)_c \cdots (t_k)_c$.

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Link with Temperley-Lieb algebras

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Let c be a Coxeter element in $\mathfrak{S}_n$, $x \in P_c$. Then

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- **Open problem**: find a combinatorial proof of the above positivity theorem.

Simple dual braids are Mikado braids

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Let W be a finite Coxeter group, c a Coxeter element, $x \in P_c$. Recall the group homomorphism $\varphi : B(W) \to \mathcal{H}(W)^\times$.

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Proof.

We have seen that x_c is Mikado. By Dyer's positivity, the result follows. □

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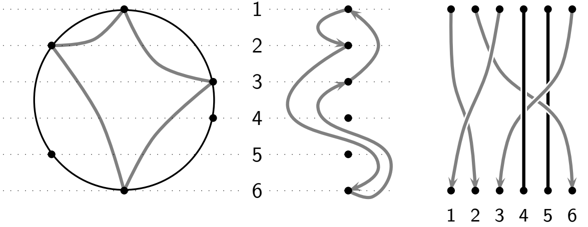

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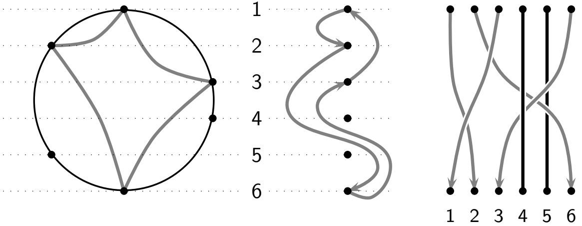

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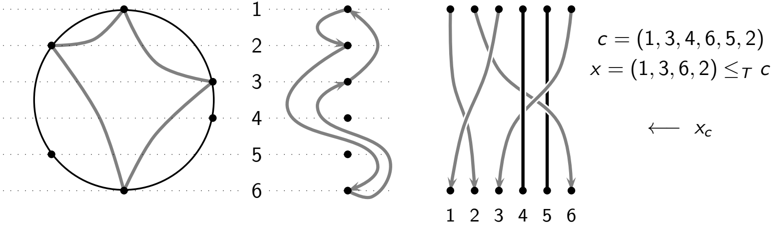

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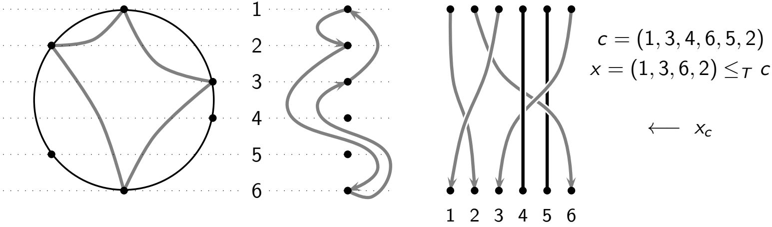

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A few remarks

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- The positive canonical lifts of elements of W have images which yield a basis of $\mathcal{H}(W)$ (the standard basis).

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- In type D_n, there are more simple elements than the dimension of the Temperley-Lieb algebra.

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- Let $x \in P_c \subseteq \mathfrak{S}_n$.

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Positivity in Temperley-Lieb algebra

Open questions

- Is there a Temperley-Lieb like algebra of type D_n, which has as basis the images of the simple dual braids ? More generally, can we use the dual approach to generalize Temperley-Lieb algebras ?
- Is there a categorification of the dual braid monoid ?
- Do the images of the simple dual braids in the Temperley-Lieb algebras of types B_n or D_n have nice positivity properties ?
- There are (quasi-)Garside structures on $B(W)$ in case W has type $\widetilde{A}_n$ or $\widetilde{C}_n$ (warning: only for certain choices of Coxeter element). Can we show that simple dual braids are Mikado braids in these cases ?
- Let $x \in P_c \subseteq \mathfrak{S}_n$. Since x_c is Mikado, there is a (in general not unique) $y \in W$ such that $x_c = x_{N(y)}$.

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THANK YOU !