Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

I. Mikado braids

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Junior Hausdorff Trimester
"Symplectic Geometry and Representation Theory"
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Classical braid group

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- Artin braid group on n strands: $\mathcal{B}_n :=$

$$\left\langle \mathbf{s}_1, \ldots, \mathbf{s}_{n-1} \;\middle|\; \begin{array}{cc} \mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_i = \mathbf{s}_{i+1}\mathbf{s}_i\mathbf{s}_{i+1} & i \leq n-2, \\ \mathbf{s}_i\mathbf{s}_j = \mathbf{s}_j\mathbf{s}_i & |i-j| > 1. \end{array} \right\rangle$$

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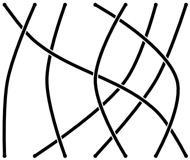

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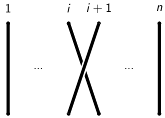

$=: \mathbf{s}_i.$

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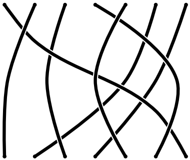
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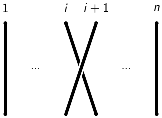
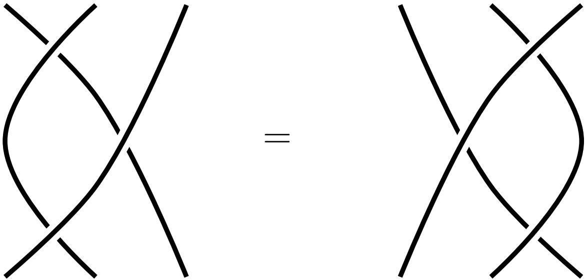


$=: \mathbf{s}_i$.

- Symmetric group on n letters: $\mathfrak{S}_n :=$

$$\left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{cc} s_i^2 = 1 & \forall i \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} & i \leq n-2, \\ s_is_j = s_js_i & |i-j| > 1. \end{array} \right\rangle$$

Braid relation: some topological observations



$$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1 = \mathbf{s}_2\mathbf{s}_1\mathbf{s}_2$$

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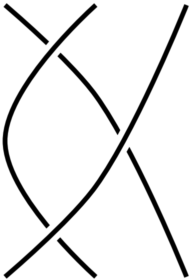

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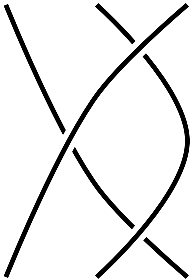

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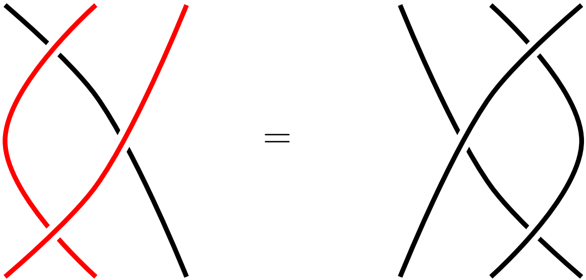

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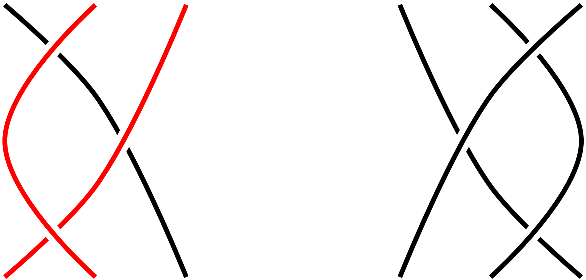

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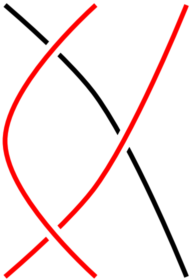

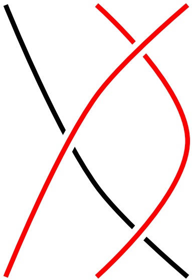

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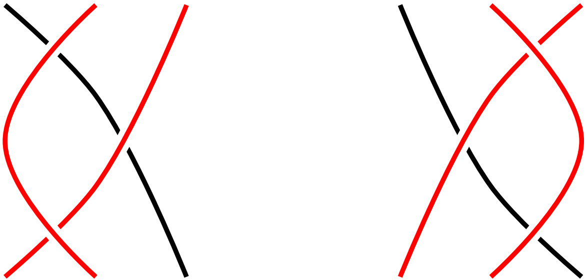

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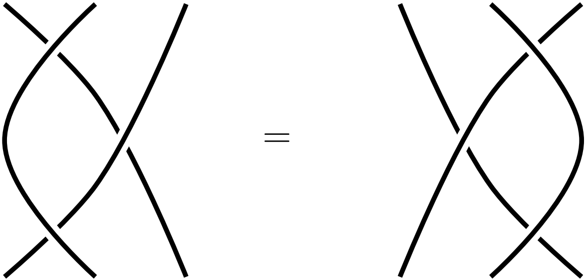

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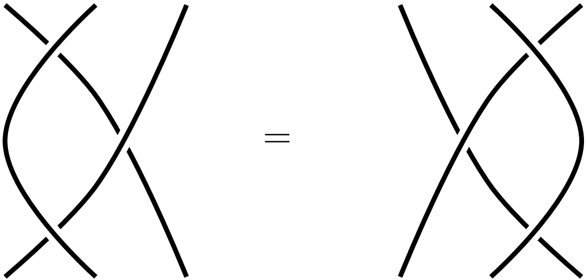

$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1^{-1} = \mathbf{s}_2^{-1}\mathbf{s}_1\mathbf{s}_2$ “Mixed” braid relation

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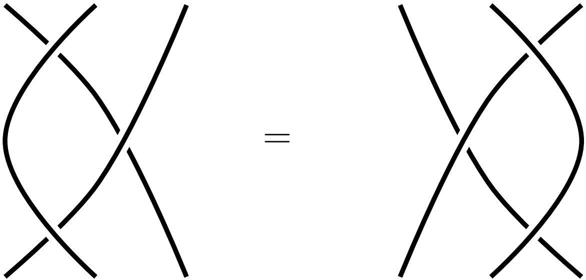

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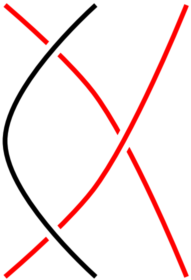

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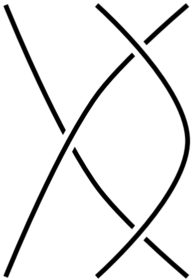

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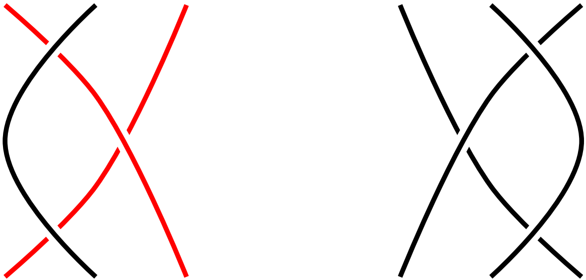

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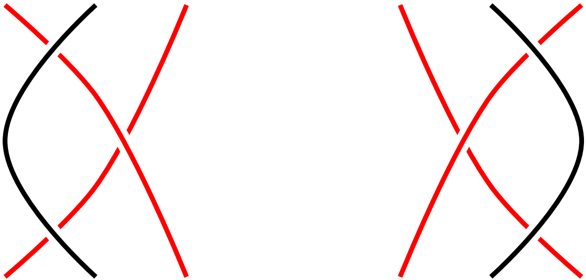

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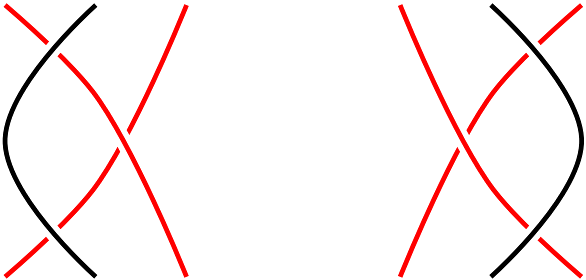

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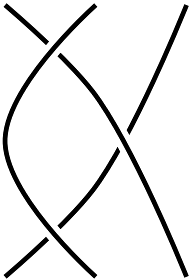

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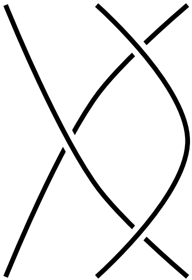

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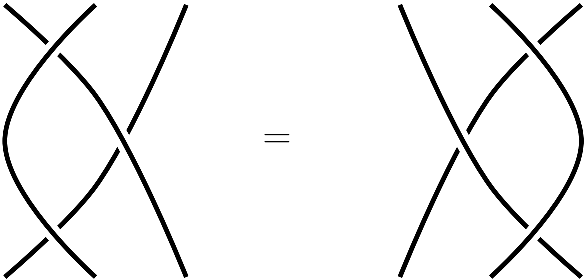

$$\mathbf{s}_1\mathbf{s}_2^{-1}\mathbf{s}_1^{-1} = \mathbf{s}_2^{-1}\mathbf{s}_1^{-1}\mathbf{s}_2$$

"Mixed" braid relation

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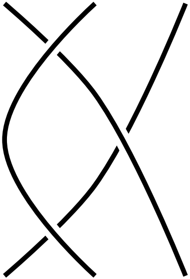

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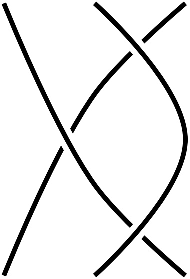

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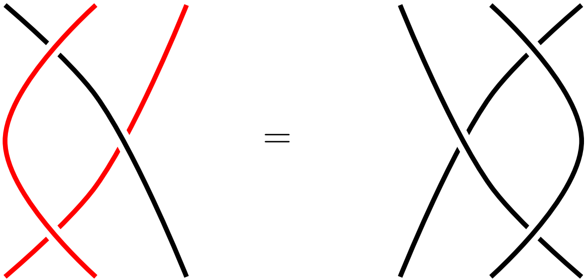

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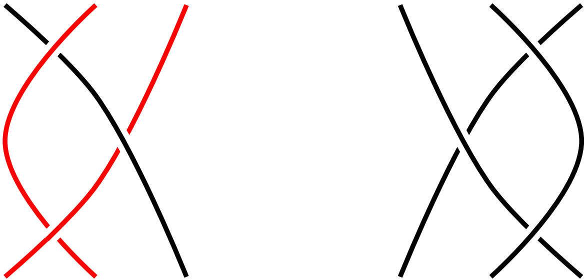

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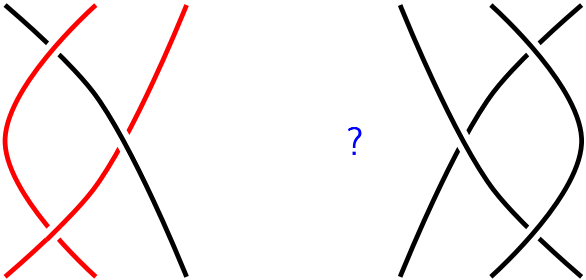

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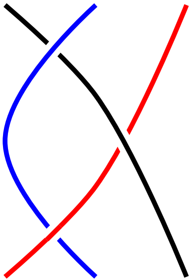

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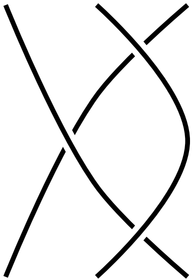

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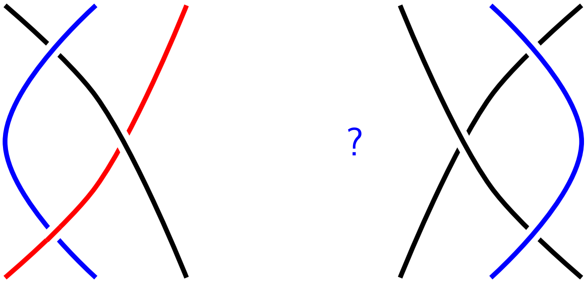

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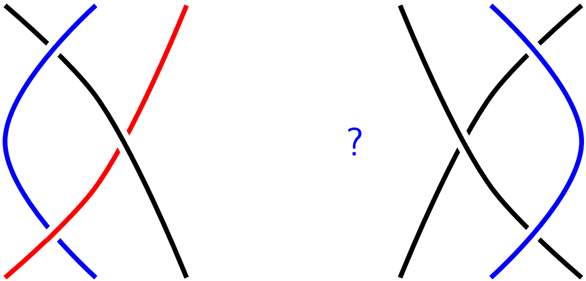

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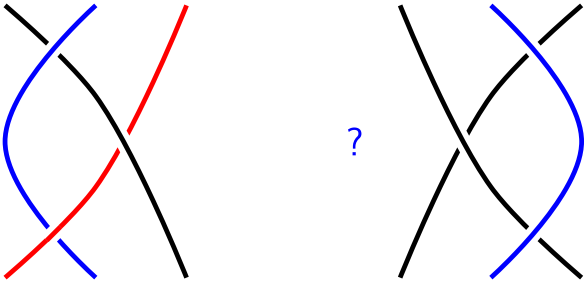

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$\Rightarrow$ Obstruction to a mixed braid relation.

Mixed braid relations

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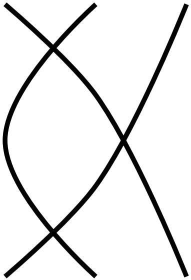

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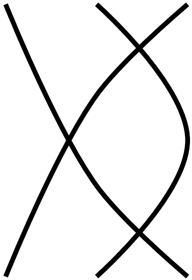

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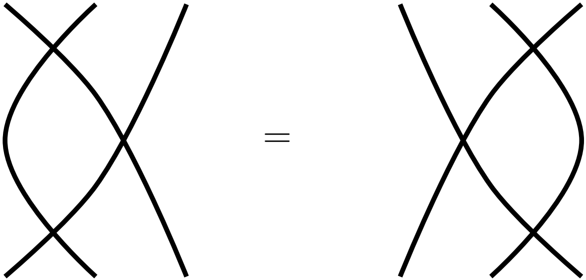
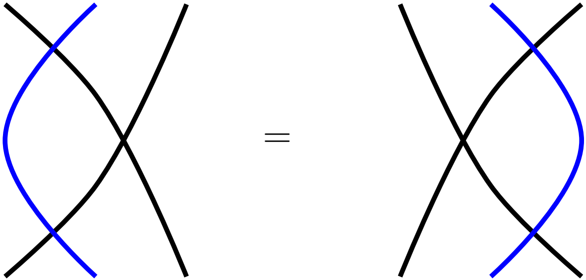


- What is the condition to put on the crossings / strands in the left braid above so that there exists a mixed braid relation?

Mixed braid relations



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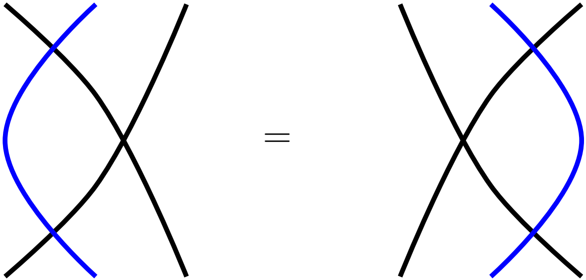

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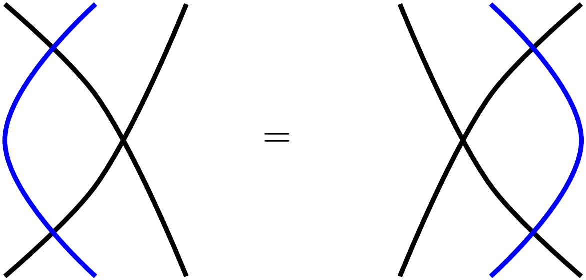

- What is the condition to put on the crossings / strands in the left braid above so that there exists a mixed braid relation?
- The blue strand should be either "above", "below" the crossing or "in between" the other two strands.
- In other words: you can remove all the strands of the braid, beginning by a strand which is above all the other strands, and going on in the same way.

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Definition (Mikado braids)

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- If $\beta \in \mathcal{B}_{n+1}$ is Mikado, then it can be shown that removing *any* strand lying above all the others in *any* braid diagram for β will yield a Mikado braid in $\mathcal{B}_n$.

Mikado braid: example

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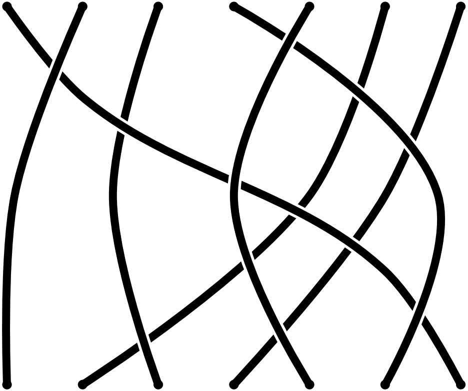

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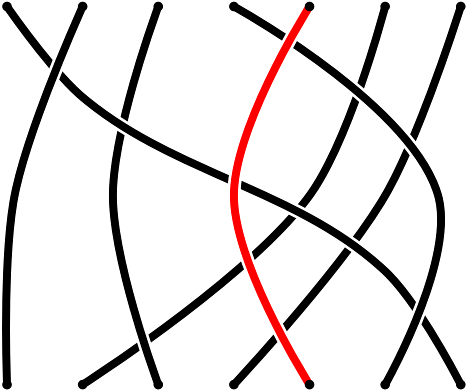

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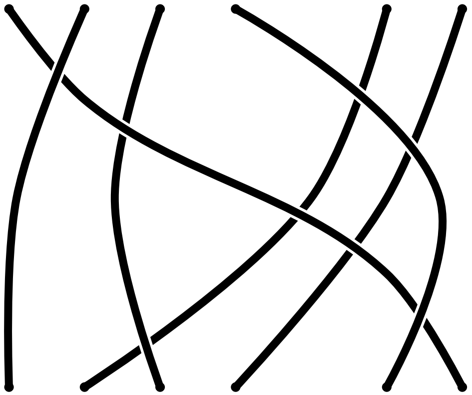

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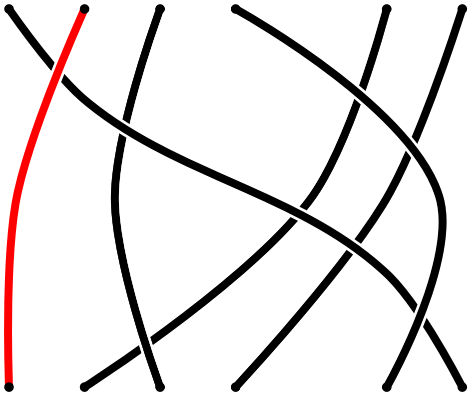

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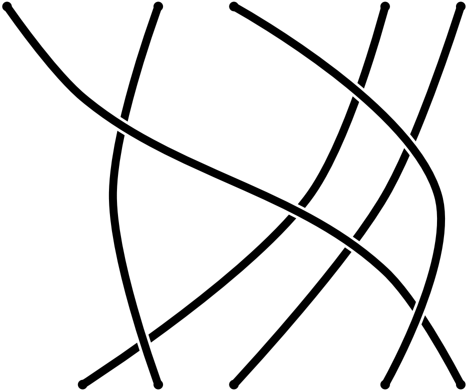

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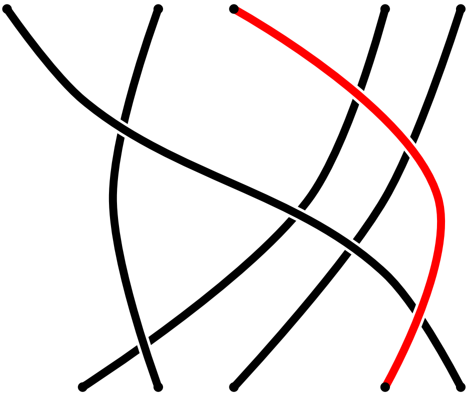

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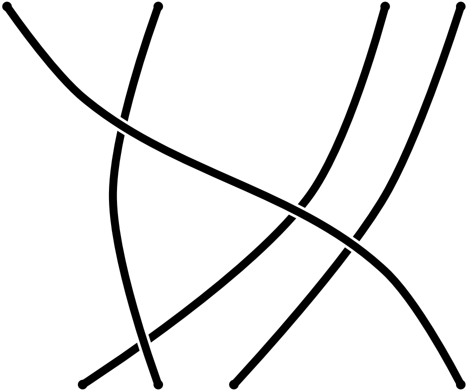

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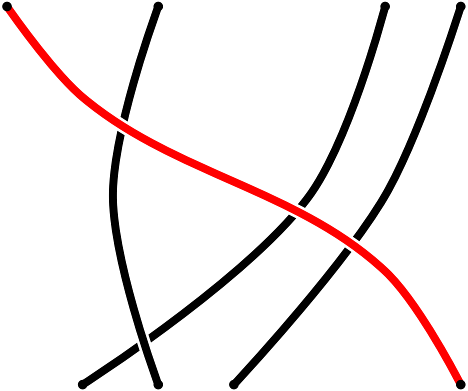

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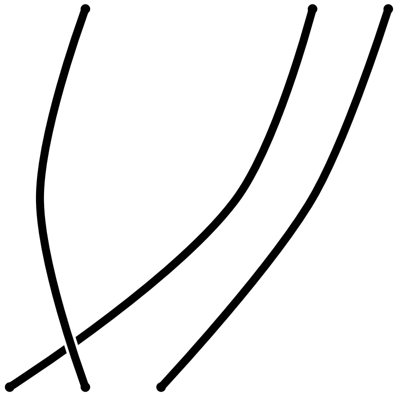

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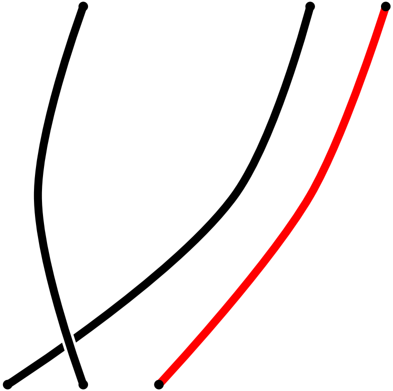

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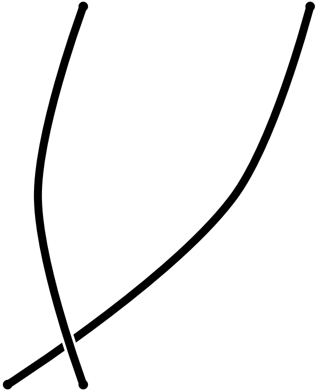

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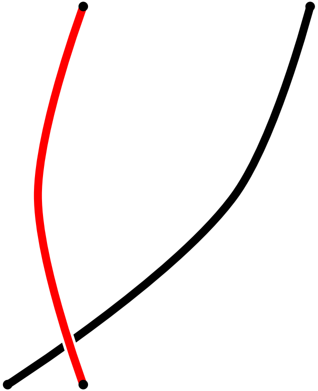

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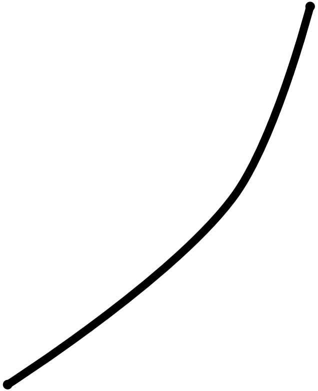

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Example: positive simple braids

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- Given a permutation $x \in \mathfrak{S}_n$, we say that a product $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $i_1, i_2, \ldots, i_k \in \{1, \ldots, n-1\}$, is a *reduced expression* of x if $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ and k is minimal. The integer $\ell(x) := k$ is the *length* of x.

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Lemma (Matsumoto's Lemma)

In the symmetric group, one can pass from any reduced expression of an element to any other by applying a sequence of braid relations.

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In the symmetric group, one can pass from any reduced expression of an element to any other by applying a sequence of braid relations.

- Every permutation $x \in \mathfrak{S}_n$ can be lifted to a *positive simple braid* (aka *canonical lift*) $\mathbf{x}$: choose a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ of x in the generating set S, then replace each s_{i_j} by $\mathbf{s}_{i_j}$.

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- Given a permutation $x \in \mathfrak{S}_n$, we say that a product $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $i_1, i_2, \ldots, i_k \in \{1, \ldots, n-1\}$, is a *reduced expression* of x if $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ and k is minimal. The integer $\ell(x) := k$ is the *length* of x.

Lemma (Matsumoto's Lemma)

In the symmetric group, one can pass from any reduced expression of an element to any other by applying a sequence of braid relations.

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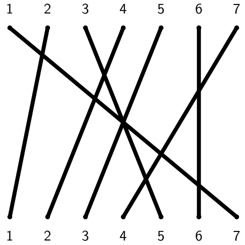

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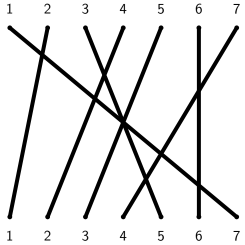

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- The number of crossings in D_x (counted with multiplicities !) is equal to $\ell(x)$ (in the example above we have $\ell(x) = 10$).

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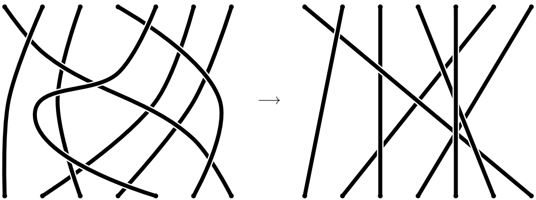

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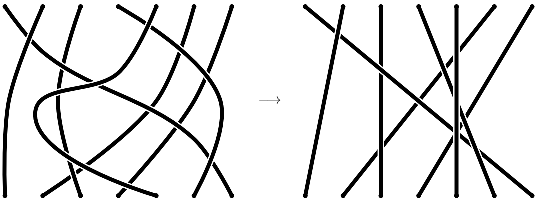

- This means that a Mikado braid can be obtained by "lifting a reduced expression of a permutation", i.e., replacing each generator s in the expression by $\mathbf{s}^{\pm 1}$.

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Definition (Square-free braids)

Let $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of a permutation. A braid of the form $\mathbf{s}_{i_1}^{\varepsilon_1} \mathbf{s}_{i_2}^{\varepsilon_2} \cdots \mathbf{s}_{i_k}^{\varepsilon_k}$ where $\varepsilon_j \in \{\pm 1\}$ for all j is a *square-free braid*.

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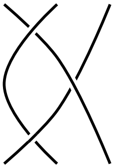

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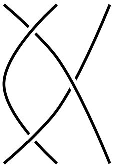

- **Question:** Can we characterize those square-free braids which are Mikado?

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- Starting from a longer reduced expression, for example $s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 s_1$, is there a way to determine which lifts are Mikado?

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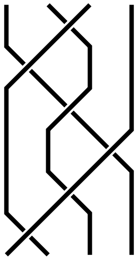

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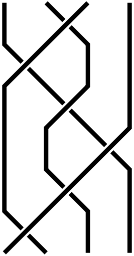

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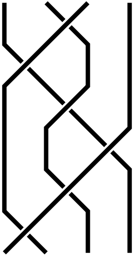

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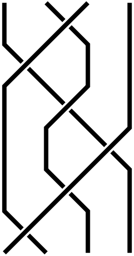

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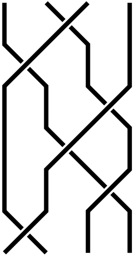

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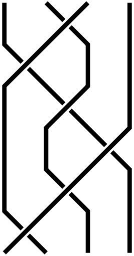

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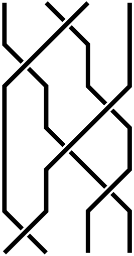

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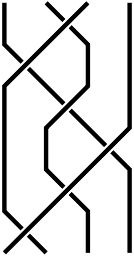

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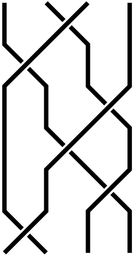

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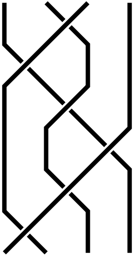

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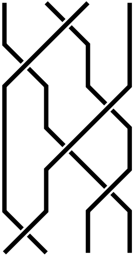

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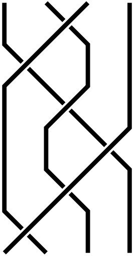

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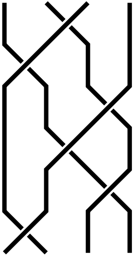

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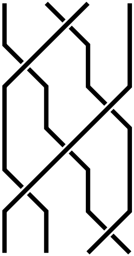

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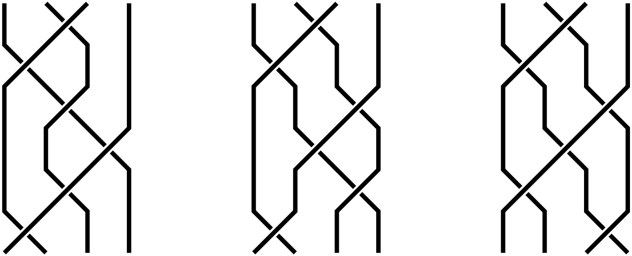

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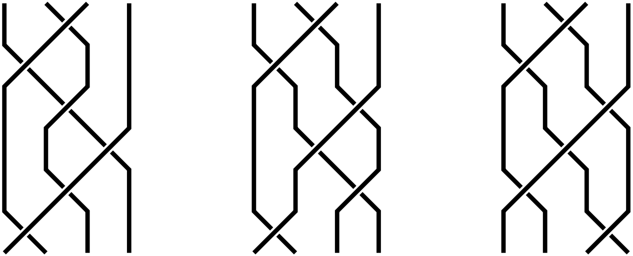

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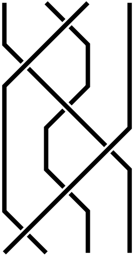

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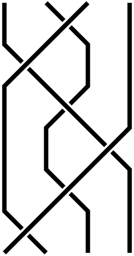

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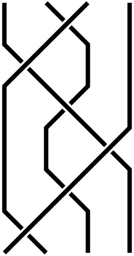

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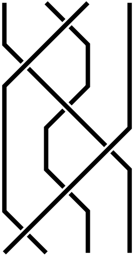

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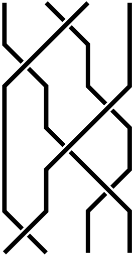

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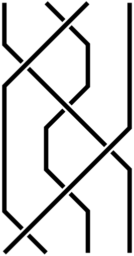

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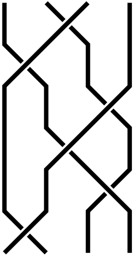

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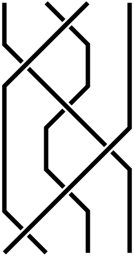

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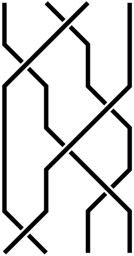

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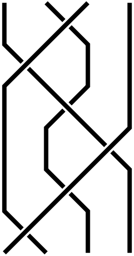

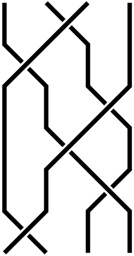

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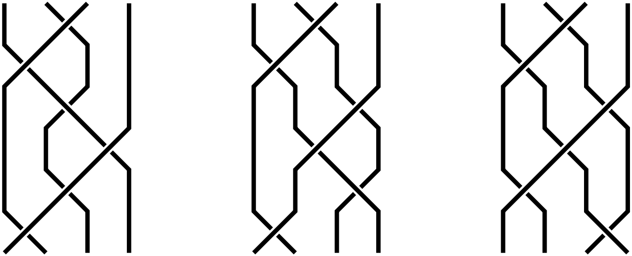

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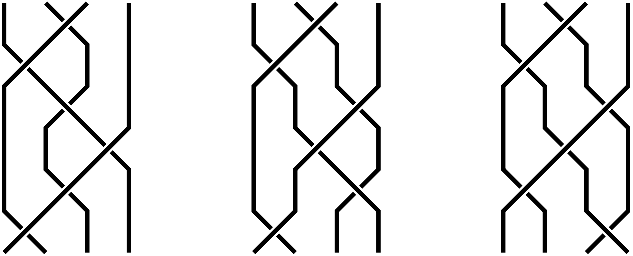

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Proposition

Let $s_{i_1}s_{i_2}\cdots s_{i_k}$ *be a reduced expression of* $x \in \mathfrak{S}_n$*. Let* $\beta = \mathbf{s}_{i_1}^{\varepsilon_1}\mathbf{s}_{i_2}^{\varepsilon_2}\cdots\mathbf{s}_{i_k}^{\varepsilon_k}$, $\varepsilon_j = \pm 1$, *be a lift of* x*. Assume that* β *is Mikado. There is a bijection between the reduced expressions of* x *and those of* β*;*

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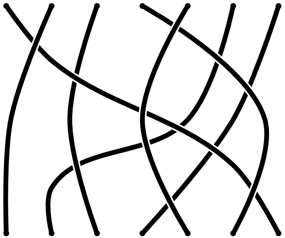

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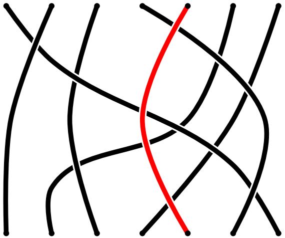

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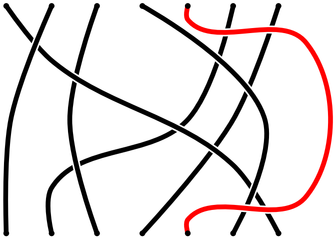

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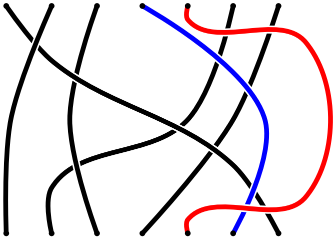

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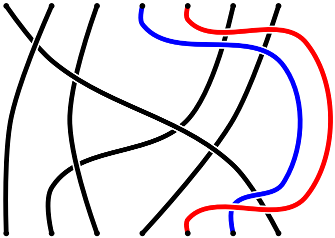

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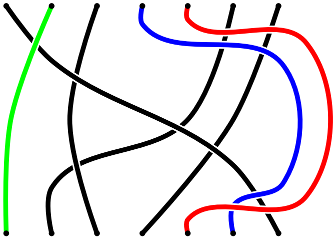

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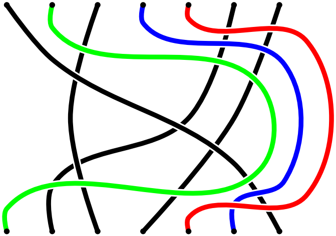

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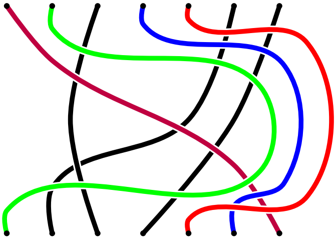

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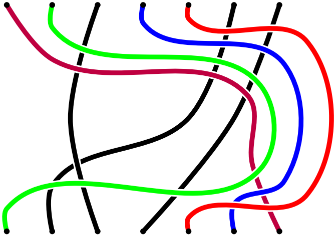

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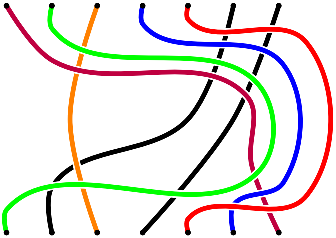

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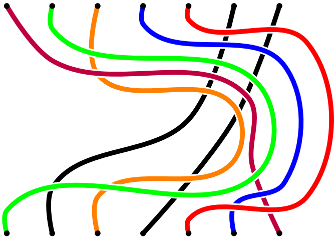

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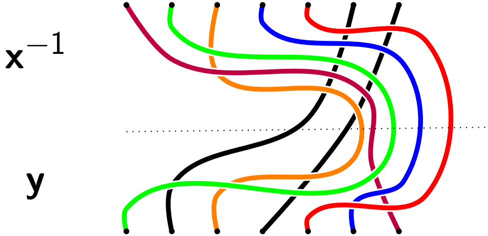

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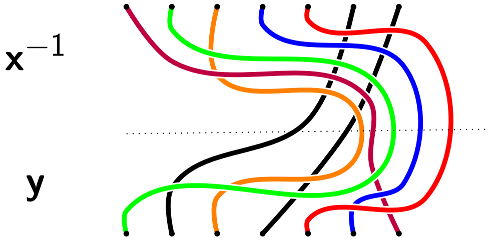

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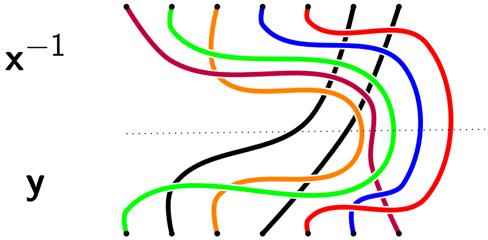

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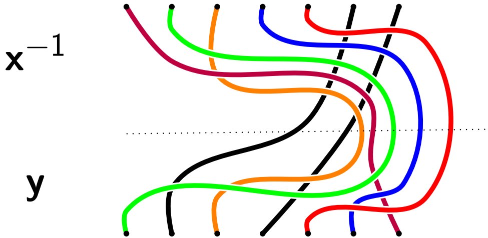

Hence we get:

Proposition

A braid $\beta \in \mathcal{B}_n$ is a Mikado braid if and only if there are $x, y \in \mathfrak{S}_n$ such that $\beta = \mathbf{x}^{-1}\mathbf{y}$, if and only if there are $u, v \in \mathfrak{S}_n$ such that $\beta = \mathbf{u}\mathbf{v}^{-1}$.

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$$x_{N(y)} := \mathbf{s}_{i_1}^{\varepsilon_1} \mathbf{s}_{i_2}^{\varepsilon_2} \cdots \mathbf{s}_{i_k}^{\varepsilon_k},$$

where $\varepsilon_j = -1$ if $s_{i_k} s_{i_{k-1}} \cdots s_{i_j} \cdots s_{i_{k-1}} s_{i_k} \in N(y)$ and 1 otherwise.

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Corollary

A braid $\beta \in \mathcal{B}_n$ is Mikado iff there are $x, y \in \mathfrak{S}_n$ such that $\beta = x_{N(y)}$.

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Symmetric group = Coxeter group

Braid group = Artin-Tits group

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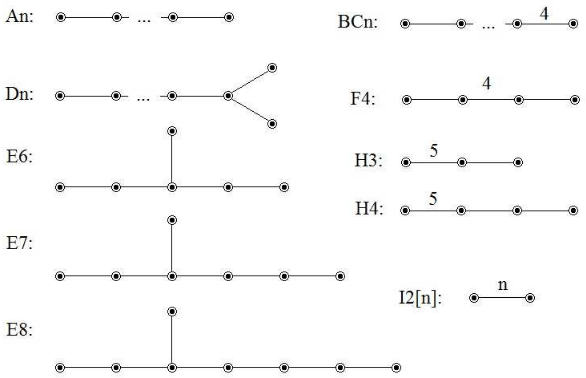
- For reasons which will become clear later, for the moment we do *not* want to restrict to finite Coxeter groups (equivalently spherical Artin-Tits groups).

Finite Coxeter groups



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- **Question:** Can we define a Mikado braid in a general Artin-Tits group?

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- In particular, if W is finite, then biclosed sets of roots are precisely inversion sets of elements.
- **Exercise:** Let W be the infinite dihedral group (i.e. $|S| = 2$, no braid relation). Show that the biclosed sets of roots are exactly inversion sets of elements, their complements, plus two infinite sets of roots which are complement to each other.

Mikado braids in general Artin-Tits groups

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Definition (Mikado braids in Artin-Tits groups)

Let W be a Coxeter group. We say that $\beta \in B(W)$ is a *Mikado braid* if there is $x \in W$ and a biclosed set $A \subseteq \Phi^+$ such that $\beta = x_A$.

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- The topological definition of Mikado braids is, as we will see further, useful and even necessary in some cases to show results involving Mikado braids.
- **Question:** Is there a topological characterization of Mikado braids in the above mentioned cases?

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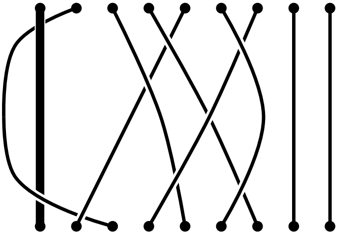
First model: Artin braids on $n+1$ strands with an unbraided first strand.



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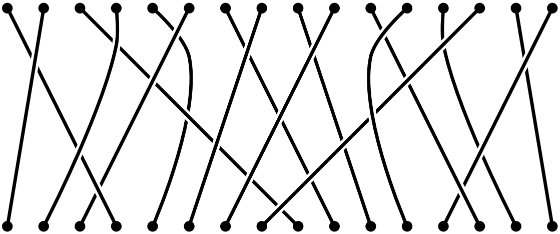
Second model: symmetric braids on 2n strands



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Let $\beta \in B(W^\Gamma)$. The following are equivalent

1. *The braid β is a Mikado braid.*
2. *There is an Artin braid in $B(W)$ representing β, such that one can inductively remove pairs of symmetric strands, one of the two strands being above all the other strands (so that the symmetric one is under all the other strands).*

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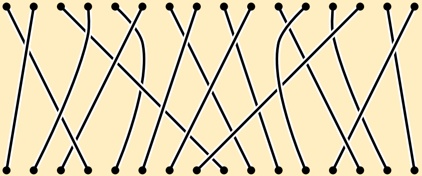
Example (A Mikado braid in type B_8)

Second model: symmetric braids on $2n$ strands



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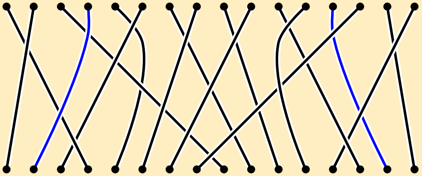

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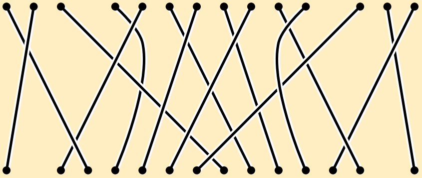

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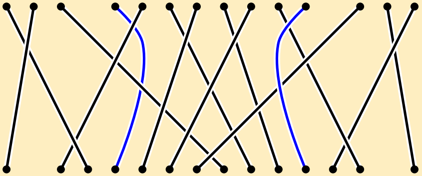

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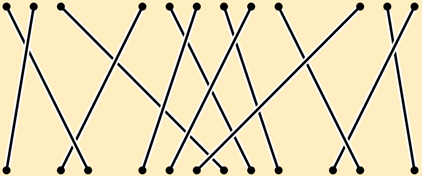

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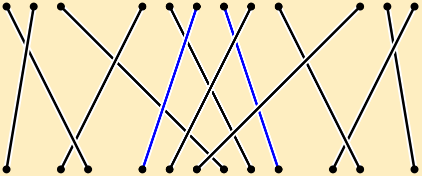

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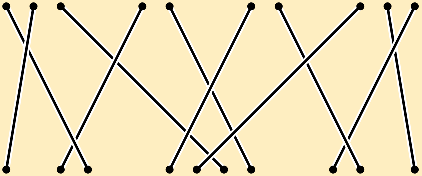

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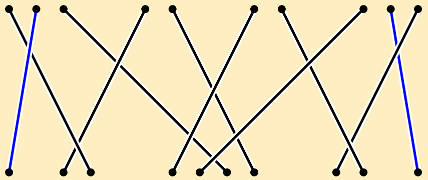

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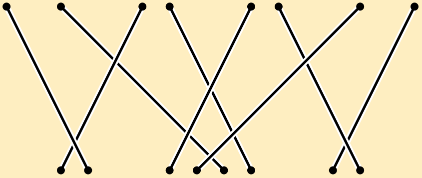

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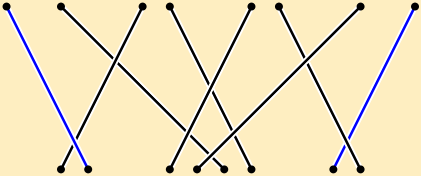

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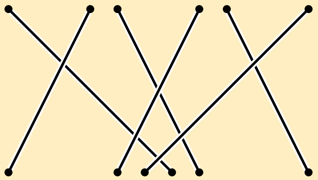

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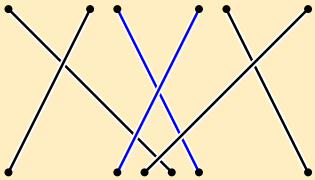

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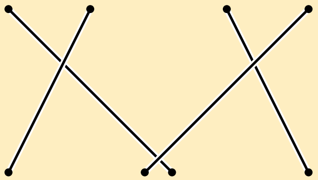

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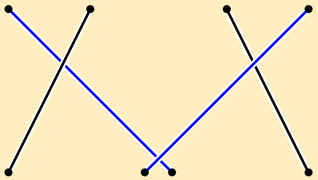

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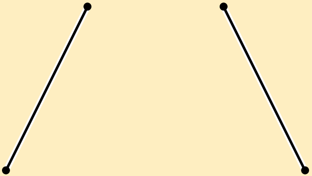

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Theorem (Baumeister-G. 2017)

Let $\beta \in B(W') \subseteq \overline{B}$. The following are equivalent.

1. *The braid β is Mikado.*
2. *There is a Mikado braid $\beta' \in B(W^\Gamma)$ such that $\beta = \pi(\beta')$, where $\pi : B(W^\Gamma) \twoheadrightarrow \overline{B}$ is the quotient map.*

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  - ... maybe some others ?
- **Open question:** is there a topological interpretation of Mikado braids in these cases?

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