HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS

UNIVERSITÄT BONN

Workshop on

"Tropical geometry and the geometry of linear programming"

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organized by
Xavier Allamigeon (INRIA IP Paris), Jesús De Loera (UC Davis), Laura Sanità (TU Eindhoven), László Végh (LSE London)

Abstracts

Plenary talks

Daniel Dadush (CWI Amsterdam)

Probabilistic analysis of the simplex method and polytope diameter

Abstract: In this talk, I will overview progress in our probabilistic understanding of the (shadow vertex) simplex method in three different settings: smoothed polytopes (whose data is randomly perturbed), well-conditioned polytopes (e.g., TU systems), and random polytopes with constraints drawn uniformly from the sphere. I will first highlight improvements and simplifications in the complexity analysis of solving smoothed linear programs via the simplex method, whose study was initiated by Spielman and Teng (JACM 04). Secondly, I will tackle the (harder) question of polytope diameter. I will explain how randomly chosen shadow paths yield the best known diameter bounds for well-conditioned polytopes, and nearly tight diameter bounds (up to one factor of dimension) for random spherical polytopes when the number of constraints is (very) large compared to dimension. This is based on joint works with Sophie Huiberts, Nicolai Hähnle, Gilles Bonnet, Uri Grupel and Galyna Livshyts.
In terms of open problems, apart from the many sub-optimal dependencies, I would like to highlight two more general questions. Firstly, can we quantify why the simplex method is so effective at re-optimizing after adding additional constraints? This is the main reason simplex is dominant in practice. Secondly, can smoothed analysis be used to explain the unreasonably low iteration counts needed to solve most LPs via interior point methods (50-100 independent of dimension!)?

Georg Loho (University of Twente)

Oriented matroids and signed tropical convexity

Abstract: Tropical convexity is the analog of classical convexity over the tropical algebra which has max and plus as its basic operations. However, tropical convexity inherently has only non-negative objects. Resolving this restriction leads to the introduction of signed tropical numbers.

They allow to formulate signed tropical convexity, which arises in different flavours. I will survey basic properties of two notions of signed tropical convexity, TO-convexity and TC-convexity, which are based on open and closed signed tropical halfspaces, respectively. While this sheds new light on the correspondence with mean payoff games and linear programming, many (complexity) questions remain open.
Oriented matroids capture the combinatorial essence of linear programming and separation by hyperplanes in general. It provides a general framework for understanding the interplay between linear programming, in particular the simplex method, and tropical convexity. I will introduce it including further potential directions.
We will also come across matroids over hyperfields and real Bergman fans along the way. It turns out that oriented matroids itself can be studied via signed tropical convexity.

Further reading:

- Tropical aspects of linear programming
  Pascal Benchimol, Theses, École Polytechnique, 2014
- Signed tropical convexity
  Georg Loho, László A. Végh, ITCS 2020
- Oriented matroids from triangulations of products of simplices
  Marcel Celaya, Georg Loho, Chi Ho Yuen, 2020, preprint arXiv:2005.01787.

Noriyoshi Sukegawa (Tokyo University of Science)

On the diameter of polyhedra and related topics

Abstract: Many questions about the behavior of the simplex method remain open today. The (combinatorial) diameter of polyhedra provides a lower bound on the worst-case complexity of the simplex method. The aim of this talk is to overview previous results on the diameter of polyhedra from various perspectives and point up some open questions with emphasis on our recent results with Antoine Deza, Shinji Mizuno, and Lionel Pournin.

Cynthia Vinzant (University of Washington)

Log concave polynomials and matroids

Abstract: Strong log concavity is a functional property of a real multivariate polynomial that translates to useful conditions on its coefficients and features in the polynomials defining several common conic programs. Recent work by several independent authors shows that the multivariate basis-generating polynomial of a matroid is strongly log concave as a function on the positive orthant. Consequences include the log concavity of face numbers of the independence complex and fast mixing times for random walks on its faces. Brändén and Huh further show that tropicalizations these polynomials coincide with M-concave functions. In this talk, I will introduce the underlying combinatorial and geometric structure of this class of polynomials and some recent applications. This is based on joint work with Nima Anari, Kuikui Liu, and Shayan Oveis Gharan.

Contributed talks

Marianne Akian (Inria and CMAP, École polytechnique CNRS, IP Paris)

Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria

Abstract: We study a tropical linear regression problem consisting in finding the best approximation of a set of points by a tropical hyperplane. We establish a strong duality theorem, showing that the value of this problem coincides with the maximal radius of a Hilbert's ball included in a tropical polyhedron. We also show that this regression problem is polynomial-time equivalent to mean payoff games. We also study variants of the tropical linear regression problem, involving the signed notion of tropical hyperplane.
We illustrate our results by solving an inverse problem from auction theory. In this setting, a tropical hyperplane represents the set of equilibrium prices. Tropical linear regression allows us to quantify the distance of a market to the set of equilibria, and infer secret preferences of a decision maker.
This talk is based on a joint work with Stephane Gaubert, Yang Qi, and Omar Saadi. See arXiv:2106.01930.

Alexander Black (UC Davis)

Modifications of the Shadow Vertex Pivot Rule

Abstract: The shadow vertex pivot rule is a fundamental tool for the probabilistic analysis of the Simplex method initiated by Borgwardt in the 1980s. More recently, the smoothed analysis of the Simplex method first done by Spielman and improved upon by Dadush and Huiberts relied on the shadow vertex pivot rule. One may also use this pivot rule to provide bounds on the diameters of polytopes. For example, Dadush and Hähnle argued that the shadow vertex pivot rule takes polynomially many steps for linear programs with totally unimodular constraint matrices. All of these results are grounded in probabilistic arguments.
In this talk, I will introduce some novel modifications to this pivot rule that make its analysis amenable to deterministic arguments. In particular, I will bound the number of non-degenerate steps the Simplex method with modified shadow vertex pivot rules takes for solving LPs on certain well known classes of polytopes. Some of the new results presented are from joint work with J. De Loera, S. Kafer and L. Sanità.

Steffen Borgwardt (UC Denver)

The role of partition polytopes in data analysis

Abstract: The field of optimization, and polyhedral theory in particular, provides a powerful point of view on common tasks in data analysis. In this talk, we highlight the role of the so-called partition polytopes and their studies in clustering and classification. The geometric properties of partition polytopes, such as their skeleton or their set of circuits, encode useful information on the underlying data set, and clusterings thereof. While a challenge in working with these polytopes lies in the lack of an explicit H-representation, they are linear projections of classical transportation polytopes, which allows a transfer of the geometric properties to algorithmic use. We cover some examples such as balanced least-squares assignments, classifiers in the form of generalized Voronoi diagrams, a measure for the robustness of a clustering, and the separation-preserving transformation of clusterings.

Francisco Criado (TU Berlin)

The dual 1-fair packing problem and applications to linear programming

Abstract: Proportional fairness (also known as 1-fairness) is a fairness scheme for the resource allocation problem introduced by Nash in 1950. Under this scheme, an allocation for two players is unfair if a small transfer of resources between two players results in a proportional increase in the utility of one player larger than the proportional decrease in the utility of the other player. For multiple players, the proportional fair resource allocation is the allocation maximizing the sum of the logarithms of the player's utilities.
In several real life applications, for example, network flows, the feasible region is the positive quadrant limited by positive linear constraints. This problem is known as the *1-fair packing problem.*
In this talk we introduce the study of its Lagrange dual. This dual problem appears naturally in the analysis of an old linear programming algorithm, Yamnitsky and Levin's "simplices" algorithm. We will show how to use the geometric relationship between primal and dual problems to find a fast algorithm for the dual, and how this algorithm could improve on Yamnitsky and Levin's algorithm.
This is a joint work with Prof. Dr. Sebastian Pokutta (ZIB) and David Martínez (University of Oxford)

Alberto Del Pia (University of Wisconsin-Madison)

Proximity in concave integer quadratic programming

Abstract: A classic result by Cook, Gerards, Schrijver, and Tardos provides an upper bound of $n\Delta$ on the proximity of optimal solutions of an Integer Linear Programming problem and its standard linear relaxation. In this bound, n is the number of variables and Δ denotes the maximum of the absolute values of the subdeterminants of the constraint matrix. Hochbaum and Shanthikumar, and Werman and Magagnosc showed that the same upper bound is valid if a more general convex function is minimized, instead of a linear function. No proximity result of this type is known when the objective function is nonconvex. In fact, if we minimize a concave quadratic, no upper bound can be given as a function of n and Δ. Our key observation is that, in this setting, proximity phenomena still occur, but only if we consider also approximate solutions instead of optimal solutions only. In our main result we provide upper bounds on the distance between approximate (resp., optimal) solutions to a Concave Integer Quadratic Programming problem and optimal (resp., approximate) solutions of its continuous relaxation. Our bounds are functions of n, Δ and a parameter ε that controls the quality of the approximation. Furthermore, we discuss how far from optimal are our proximity bounds. This is joint work with Mingchen Ma.

Nathanaël Fijalkow (CNRS LaBRI and Alan Turing Institute)

Understanding and extending the quasipolynomial time algorithms for parity games

Abstract: This talk is about the model of two-player (deterministic) parity games, their extensions mean payoff games, and related game models. The dust has settled since the 2017 breakthrough—a quasipolynomial time algorithm for solving parity games. A lot of work has gone since then into understanding the original algorithm and constructing new ones with the same complexity. The first point of this talk is to argue that all existing algorithms for parity games are tightly connected to the (purely combinatorial) notion of universal trees. The second point is that universal trees, and more precisely their extensions to any positionally determined objectives called universal graphs, are a new and very powerful tool for constructing algorithms for solving games. Time permitting, I will discuss the recent results in the area: (partial) complexity lower bounds, new algorithms for mean payoff games and beyond, and tropical interpretations of universal graphs.

Stephane Gaubert (INRIA and CMAP, École Polytechnique, IP Paris, CNRS)

Tropical convexity and its relation with mean payoff games and linear programming

Abstract: Convex sets can be defined over ordered fields with a non-archimedean valuation. Then, tropical convex sets arise as images by the valuation of non-archimedean convex sets. The tropicalizations of polyhedra and spectrahedra are of special interest, since they can be described in terms of deterministic and stochastic games with mean payoff. In that way, one gets a correspondence between classes of zero-sum games, with an unsettled complexity, and classes of semialgebraic convex optimization problems over non-archimedean fields, including linear and semidefinite programming [1,3]. Recently, this correspondence led to a counter example showing that log-barrier interior point methods are not strongly polynomial [2]. I will give a general overview of results concerning the interplay between tropical convexity and zero-sum mean payoff games. I will conclude with a recent development, on ambitropical convexity (which encompasses tropical convexity and its dual in a unified framework) [4].
This survey is based on a series of works with different coauthors, especially,

[1] Tropical polyhedra are equivalent to mean payoff games, M. Akian, S. Gaubert, A. Guterman, International of Algebra and Computation, 22(1):125001, (2012)

[2] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig, What Tropical Geometry Tells Us about the Complexity of Linear Programming, SIAM review, 63(1), 123–164 (2021).

[3] X. Allamigeon, S. Gaubert, and M. Skomra. Tropical spectrahedra, Discrete Comput. Geom., 63, 507–548 (2020).

[4] M. Akian, S. Gaubert, S. Vannucci, Ambitropical convexity: The geometry of fixed point sets of Shapley operators, preprint (2021).

Michael Joswig (TU Berlin)

Generalized permutahedra and optimal auctions

Abstract: We study a family of convex polytopes, called SIM-bodies, which were introduced by Giannakopoulos and Koutsoupias (2018) to analyze so-called Straight-Jacket Auctions. First, we show that the SIM-bodies belong to the class of generalized permutahedra. Second, we prove an optimality result for the Straight-Jacket Auctions among certain deterministic auctions. Third, we employ computer algebra methods and mathematical software to explicitly determine optimal prices and revenues. The SIM-bodies arise naturally as regions of linearity of a tropical polynomial.
Joint work with Max Klimm and Sylvain Spitz.

Sean Kafer (University of Waterloo)

Performance of steepest descent in 0/1 LPs

Abstract: Even after decades of study, it is unknown whether there exists a pivot rule for the Simplex method that always solves an LP with only a polynomial number of pivots. This remains unknown even in the special case of 0/1 LPs, a case that includes many extensively studied problems in combinatorial optimization.
In the pursuit of better understanding the behavior of the Simplex method on 0/1 LPs, we study the length of the monotone edge-path generated by following steepest edges, where here we determine

steepness by normalizing with the 1-norm instead of the usual 2-norm. We show that this path is of a strongly-polynomial length and that it can be computed in polynomial time. We achieve this by first considering the length of the so-called circuit-path generated by steepest descent circuit steps, where the circuits are a set of augmentation directions that generalize the set of edges. We then show that, at an extreme point solution of a 0/1 LP, a steepest descent circuit step is always equal to a steepest descent edge step, i.e., that these concepts are unified in the setting of 0/1 LPs.
Finally, in light of the fact that these results do not describe the behavior of the steepest edge Simplex pivot rule, we devise an alternate pivot rule for 0/1 LPs which is guaranteed to follow the path generated by steepest edges. That is, we show the existence of a Simplex pivot rule for 0/1 LPs which is guaranteed to require only a polynomial number of non-degenerate pivots.
Joint work with Jesús De Loera, Laura Sanità, and Alex Black.

Zhuan Khye (Cedric) Koh (LSE)

Beyond value iteration for parity games: strategy iteration with universal trees

Abstract: Parity games have witnessed several new quasi-polynomial algorithms since the breakthrough result of Calude et al. (2017). The central combinatorial object underlying these approaches is a universal tree, as identified by Czerwiński et al. (2019). By providing a quasi-polynomial lower bound on the size of universal trees, they have highlighted a barrier that must be overcome by all existing approaches to attain polynomial runtime. This is due to the existence of worst case instances which force these algorithms to explore a large portion of the tree.
As an attempt to overcome this barrier, we propose a strategy iteration framework which can be applied on any universal tree. It matches the running time of its value iteration counterparts, while allowing one to take bigger leaps in the universal tree. Value iteration—asymptotically the fastest known algorithm for parity games—is a repeated application of operators associated with arcs in the game graph to obtain the least fixed point. Our main technical contribution is an efficient method for computing the least fixed point of operators associated with arcs in a strategy subgraph. This is achieved via a careful adaptation of shortest paths algorithms to the setting of ordered trees. By plugging in the universal tree of Jurdziński and Lazić (2017), or the Strahler universal tree of Daviaud et al. (2020), we obtain instantiations of the general framework that take time $O(mn^2 \log n \log d)$ and $O(mn^2 \log^3 n \log d)$ respectively per iteration.
Based on joint work with Georg Loho.

Bento Natura (LSE)

On circuit imbalance measures and their role in circuit augmentation algorithms

Abstract: The efficiency of many algorithms for linear programs and integer programs crucially depends on condition numbers of the constraint matrix. We motivate new combinatorial condition numbers that bound the ratio of non-zero entries of support-minimal vectors in the kernel of the constraint matrix. We further relate them to existing well-studied ones, prove stronger upper bounds and review circuit diameter bounds and circuit augmentation algorithms and therefore simplex methods.

Shmuel Onn (Technion)

Sparse integer programming is FPT

Abstract: We show that sparse integer programming, in variable dimension, with linear or separable convex objective, is fixed-parameter tractable. This is a culmination of a long line of research with many colleagues. We also discuss some of the many consequences of this result, which provides a new powerful tool that we believe will be useful in parameterized complexity, and overview some related results.

Lionel Pournin (University of Paris 13)

Algorithmic, combinatorial, and geometric aspects of linear optimization

Abstract: The simplex and interior point methods are currently the most computationally successful algorithms for linear optimization. While the simplex methods follow an edge path, the interior point methods follow the central path. The algorithmic issues are closely related to the combinatorial and geometric structure of the feasible region. Focusing on the analysis of worst-case constructions leading to computationally challenging instances, we discuss connections to the largest diameter of lattice polytopes, to the complexity of convex matroid optimization, and to theoretical physics. Complexity results and open questions are also presented. This talk is based on joint work with Antoine Deza.

Raman Sanyal (Goethe-Universität Frankfurt)

Polyhedral geometry of pivot rules

Abstract: Geometrically, a linear program gives rise to a polyhedron together with an orientation of its graph. A simplex method selects a path from any given vertex to the sink and thus determines an arborescence. The centerpiece of any simplex method is the pivot rule that selects the outgoing edge for a given vertex. Changing the objective function changes the orientation and hence the arborescence. For pivot rules belonging to a certain class, we associate polytopes that capture these arborescences of a given polyhedron. This gives a geometric perspective on pivot rules that is related to shadow-vertex-rules, monotone path polytopes, and polytopes from geometric combinatorics. This is joint work with Alex Black, Jesús De Loera, and Niklas Lütjeharms.

Mateusz Skomra (LAAS CNRS)

Separation theorems in signed tropical convexities

Abstract: The max-plus semifield can be equipped with a natural notion of convexity called the "tropical convexity". This convexity has many similarities with the standard convexity over the non-negative real numbers. In particular, it has been shown that tropical polyhedra are closely related to their classical counterparts. However, the restriction to non-negative numbers implies that some properties of classical objects are lost in the tropical setting. For instance, two generic tropical lines in a real tropical plane may not intersect.
In order to avoid this problem, one can work with signed tropical numbers. Recently, a definition of convexity over the signed tropical numbers was proposed by Loho and Végh. This notion, called the TO-convexity, is based on a multi-valued addition of signed tropical numbers. In this talk, I will present a new version of signed tropical convexity, the TC-convexity, and compare it with the TO-convexity. The TC-convexity is based on a non-commutative addition of signed tropical numbers,

which generalizes the composition operation of signed sets that is used to define oriented matroids. Despite being harder to analyze, one obtains surprisingly strong structural results about this notion of convexity.
Our main result is a hyperplane separation theorem for the TC-convexity. Namely, the TC-convex hull of finitely may points coincides with the intersection of all closed tropical half-spaces that contain these points. Along the way, we characterize the signed tropical segments and hemi-spaces, give the analogues of the Carathéodory theorem and the Minkowski-Weyl theorem, improve the separation results for the TO-convexity, and prove that the TC-convex hull coincides with the TO-convex hull for any generic collection of points.
This talk is based on a joint work with Georg Loho.

Ben Smith (University of Manchester and Heilbronn Institute for Mathematical Research)

Face structures of tropical polyhedra

Abstract: Many combinatorial algorithms arise from the interplay between faces of ordinary polyhedra, therefore tropicalizing these algorithms should rely on the face structure of tropical polyhedra. While they have many nice combinatorial properties, the classical definition of a face is flawed when applied to tropical polyhedra. For example, they do not properly tile the boundary, and cannot be characterized by their defining vertices. Even worse, tropical polyhedra do not have a unique minimal halfspace description, and so the notion of a facet is not well defined.
To partially overcome these issues, we consider monomial tropical polyhedra, a class arising as the tropicalization of blocking polyhedra. Unlike general tropical polyhedra, these have a unique minimal halfspace description. Furthermore, we show that this can be extended to a canonical halfspace description for any tropical polyhedron, giving a coherent notion of tropical facet. We use this class as a jumping off point to investigate possible face structures of tropical polyhedra and their combinatorics.
This is joint work with Georg Loho.

Stefan Weltge (TU Munich)

Binary scalar products

Abstract: We settle a conjecture by Bohn, Faenza, Fiorini, Fisikopoulos, Macchia, and Pashkovich (2015) concerning 2-level polytopes. Such polytopes have the property that for every facet-defining hyperplane H there is a parallel hyperplane H' such that H and H' contain all vertices. The authors conjectured that for every d-dimensional 2-level polytope P the product of the number of vertices of P and the number of facets of P is at most $d2^{d+1}$, which we show to be true.
