On the homogenization problem for the linear Boltzmann equation

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Qualitative behaviour of kinetic equations and related problems

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Results in collaboration with H. Hutridurga and O. Mula

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The linear Boltzmann equation

Describes the behavior of a neutron gas interacting with a host medium (typically, the core of a nuclear reactor)

Ω: bounded domain of $\mathbb{R}^d$ with C^1 boundary
$\mathbb{V}$: velocity space
$f = f(t, x, v)$: population density of neutrons

$$\partial_t f + v \cdot \nabla_x f + \sigma(x, v) f - \int_{\mathbb{V}} \kappa(x, v \cdot v') f(t, x, v') \,\mathrm{d}v' = 0$$

Optical parameters:

$\sigma \geq 0$: total cross-section of the background material
$\kappa \geq 0$: scattering kernel

The energy description

$\omega = v/|v|$: trajectory angle of the neutron

$E = m|v|^2/2$: kinetic energy of the neutron (m: neutron mass)

New unknown: neutron flux (v expressed via the pair (ω, E))
$\varphi(t,x,\omega,E) = \varphi(t,x,v) := |v| f(t,x,v) \quad E \in [E_{\min}, E_{\max}]$

The linear Boltzmann equation

$$\sqrt{\frac{m}{2E}}\, \partial_t \varphi + \omega \cdot \nabla_x \varphi + \sigma\left(x,\omega,E\right)\varphi$$

$$- \int_{E_{\min}}^{E_{\max}} \int_{|\omega'|=1} \kappa\left(x, \omega\cdot\omega', E, E'\right) \varphi(x,\omega',E')\mathrm{d}\omega'\,\mathrm{d}E' = 0$$

$\varphi(0,x,\omega,E) = \varphi_{\mathrm{in}}(x,\omega,E)$

$\varphi = 0 \ \forall t, E > 0$ and for $(x,\omega) \in \Gamma_- = \left\{(x,\omega) \in \partial\Omega \times \mathbb{S}^{d-1} : n_x \cdot \omega < 0\right\}$

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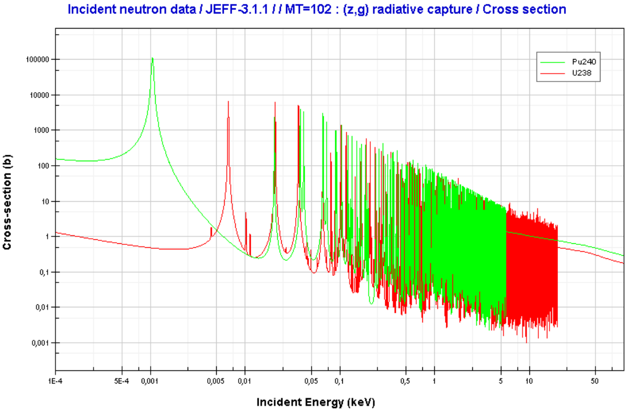
Self-shielding

In some cases the amount of absorption reactions is dramatically modified by the heterogeneity of the host medium



Radiative capture cross sections of 238U and 240Pu

Two types of self-shielding

- spatial self-shielding
- energy self-shielding

Spatial self-shielding

Basic fuel element of light water reactors: fuel rod which contains fuel pellets made of uranium dioxide.

A standard reactor core may contain some 15 million fuel pellets. The neutron moderator surrounds these fuel rods. The spatial self-shielding is phenomenon primarily connected with this heterogeneity of the reactor core.

Dumas, L. and Golse, F. Homogenization of Transport Equations. SIAM J Appl Math, 60(4), pp. 1447-1470, 2000

Energy self-shielding

Energy self-shielding is related to the high oscillation of the optical parameters with respect to the energy of the incoming flux.

Physicists noticed that the simple average of the optical parameters in the linear Boltzmann equation does not allow to obtain accurate results (reduction of the expected energy dependent neutron flux)

Practical strategy: introduce a correction on the averages of the optical parameters in the linear Boltzmann equation

H. Hutridurga, O. Mula, F. Salvarani. Homogenization in the energy variable for a neutron transport problem. Asymptotic analysis, in press

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L. Tartar's example (1979)

For the unknown u^ε, consider the differential equation

$$\partial_t u^\varepsilon + \sigma\left(\frac{x}{\epsilon}\right) u^\epsilon = 0; \qquad u^\epsilon(0,x) = u_{\rm in}(x).$$

Notation for the Laplace transform (in the time variable) of a function:

$$\widehat{f}(p) := \int_0^\infty e^{-ps} f(s)\,\mathrm{d}s \qquad \text{for } p > 0.$$

Notation: let $\mathrm{Y} := (0,1)^d$ be the unit cube in $\mathbb{R}^d$; for any $v \in \mathrm{L}^1(\mathrm{Y})$

$$\langle v \rangle = \int_{\mathrm{Y}} v(y)\,\mathrm{d}y$$

denotes the average of v in Y.

Homogenization of Tartar's example

Theorem (Tartar)

Let the coefficient $\sigma(\cdot)$ be a strictly positive, bounded and purely periodic coefficient of period Y. Then the L^∞ weak $*$ limit $u_{\mathrm{hom}}(t,x)$ of the solution family u^ϵ satisfies the following integro-differential equation

$$\begin{cases} \partial_t u_{\mathrm{hom}}(t,x) + \langle\sigma\rangle u_{\mathrm{hom}}(t,x) - \displaystyle\int_0^t \mathcal{M}(t-s) u_{\mathrm{hom}}(s,x)\,\mathrm{d}s = 0 \\ u_{\mathrm{hom}}(0,x) = u_{\mathrm{in}}(x) \end{cases}$$

where the memory kernel $\mathcal{M}(\tau)$ is given in terms of its Laplace transform

$$\widehat{\mathcal{M}}(p) = p + \langle\sigma\rangle - \mathcal{B}(p) = \int_{\mathrm{Y}} \Big(p + \sigma(y) - \mathcal{B}(p)\Big)\,\mathrm{d}y \qquad \forall p > 0,$$

with the constant $\mathcal{B}(p)$ taking the value $\mathcal{B}(p) := \left(\displaystyle\int_{\mathrm{Y}} \frac{\mathrm{d}y}{p+\sigma(y)}\right)^{-1}$.

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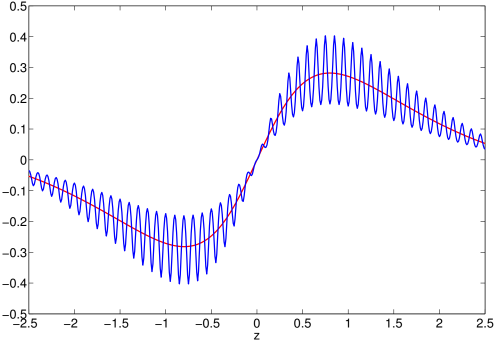
(J. Mathiaud & FS)

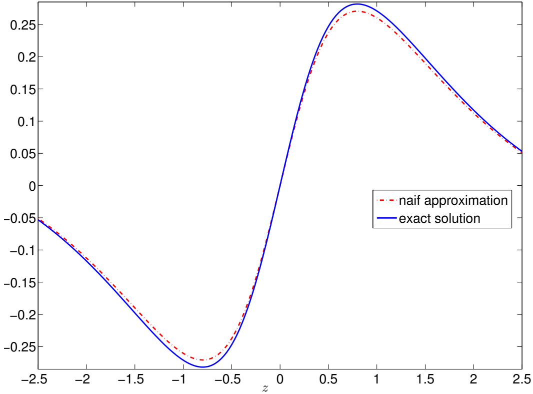


Solution of the non-homogenized problem (blue line) and of the exact homogenized problem (red line) at time $t = 0.4$

Initial condition: $u^{in}(x) = \sin(x)/(1 + x^2)$

Absorption coefficient: $\sigma = \cos(2\pi/\epsilon)/4 + 1 \qquad \epsilon = 10^{-1}$

Comparison at time $t = 0.4$



Solution of the exact homogenized problem (blue line) and solution of the problem obtained by averaging the absorption coefficient (red dotted line)

Initial condition: $u^{in}(x) = \sin(x)/(1 + x^2)$

Absorption coefficient: $\sigma = \cos(2\pi/\epsilon)/4 + 1 \qquad \epsilon = 10^{-1}$

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Tartar's example revisited

$$\begin{cases} \partial_t u^\epsilon(t,x) + \sigma^\epsilon(x) u^\epsilon(t,x) = f^\epsilon(t,x) & (t,x) \in (0,T)\times\Omega \\ u^\epsilon(0,x) = u^\epsilon_{\mathrm{in}}(x) & x \in \Omega \end{cases}$$

$$\sigma^\epsilon(x) := \sigma\left(x, \frac{x}{\epsilon}\right), \quad f^\epsilon(t,x) := f\left(t,x,\frac{x}{\epsilon}\right), \quad u^\epsilon_{\mathrm{in}}(x) := u_{\mathrm{in}}\left(x,\frac{x}{\epsilon}\right),$$

$$\sigma(x,y) \in \mathrm{L}^\infty(\Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y}))$$

$$f(t,x,y) \in \mathrm{L}^\infty((0,T)\times\Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y})), \quad u_{\mathrm{in}}(x,y) \in \mathrm{L}^2(\Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y}))$$

Notation: $\mathrm{C}_{\mathrm{per}}(\mathrm{Y})$ denote Y-periodic continuous functions on $\mathbb{R}^d$

Hypothesis: there exists a positive constant σ_{min} such that

$$\sigma(x,y) \geq \sigma_{\mathsf{min}} \quad \forall\, (x,y) \in \Omega \times \mathrm{Y}$$

Two-scale convergence

The notion of two-scale convergence is a weak-type convergence as it is given in terms of test functions

Definition

A family of functions $v^\epsilon(x) \subset \mathrm{L}^2(\Omega)$ two-scale converges to a limit $v^0(x,y) \in \mathrm{L}^2(\Omega \times \mathrm{Y})$ if, for any smooth test function $\psi(x,y)$, Y-periodic in the y variable,

$$\lim_{\epsilon \to 0} \int_\Omega v^\epsilon(x) \psi\left(x, \frac{x}{\epsilon}\right) \mathrm{d}x = \int_\Omega \int_Y v^0(x,y) \psi(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

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Two-scale convergence: two results

Theorem (Nguetseng, Allaire)

Suppose a family $v^\epsilon(x) \subset \mathrm{L}^2(\Omega)$ is uniformly bounded, i.e.,

$$\|v^\epsilon\|_{\mathrm{L}^2(\Omega)} \leq C$$

with constant C being independent of ϵ. Then, we can extract a sub-sequence (still denoted v^ϵ) such that v^ϵ two-scale converges to some limit $v^0(x, y) \in \mathrm{L}^2(\Omega \times Y)$.

Proposition (Nguetseng, Allaire)

Let v^ϵ be a sequence of functions in $\mathrm{L}^2(\Omega)$ which two-scale converges to a limit $v^0 \in \mathrm{L}^2(\Omega \times \mathrm{Y})$. Then $v^\epsilon(x)$ converges to $\langle v\rangle(x) = \int_{\mathrm{Y}} v^0(x, y)\,\mathrm{d}y$ weakly in $\mathrm{L}^2(\Omega)$, i.e.,

$$\lim_{\epsilon\to 0} \int_\Omega v^\epsilon(x)\varphi(x)\,\mathrm{d}x = \int_\Omega \varphi(x) \int_Y v^0(x, y)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{L}^2(\Omega).$$

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$$\lim_{\epsilon\to 0}\int_\Omega v^\epsilon(x)\varphi(x)\,\mathrm{d}x = \int_\Omega \varphi(x)\int_Y v^0(x,y)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{L}^2(\Omega).$$

Properties of the ODE

For any given $g \in \mathrm{L}^\infty(Y)$, the linear operator

$$\mathcal{L}_g\, v := gv - \langle gv \rangle \quad \forall v \in \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$$

is bounded in $\mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$ as

$$\|\mathcal{L}_g h\|^2_{\mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})} = \int_{\mathrm{Y}} |g(y)h(y) - \langle gh \rangle|^2 \,\mathrm{d}y = \int_{\mathrm{Y}} |g(y)h(y)|^2 \,\mathrm{d}y - \langle gh \rangle^2$$

By Cauchy-Schwarz:

$$|\langle gh \rangle| = \left| \int_{\mathrm{Y}} g(y)h(y)\,\mathrm{d}y \right| \le \left(\int_{\mathrm{Y}} |g(y)h(y)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}}.$$

As a consequence, $\mathcal{L}_g : \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y}) \mapsto \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$ is the infinitesimal generator of a uniformly continuous semigroup given by

$$e^{t\mathcal{L}_g} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_g^n.$$

Theorem (H. Hutridurga, O. Mula, FS)

$$u^\epsilon \rightharpoonup u_{\rm hom} \quad \text{weakly in } \mathrm{L}^2((0,T)\times\Omega)$$

$$\begin{cases} \partial_t u_{\rm hom}(t,x) + \langle\sigma\rangle(x) u_{\rm hom}(t,x) - \displaystyle\int_0^t \mathcal{K}(t-s,x) u_{\rm hom}(s,x)\,\mathrm{d}s = \mathcal{S}(t,x) \\ u_{\rm hom}(0,x) = \langle u_{\rm in}\rangle(x) \end{cases}$$

The memory kernel is given by

$$\mathcal{K}(\tau,x) = \int_{\mathrm{Y}} \sigma(x,y) e^{-\tau\mathcal{L}_\sigma} \mathcal{L}_1 \sigma(x,y)\,\mathrm{d}y$$

The source term is given by

$$\mathcal{S}(t,x) = \langle f\rangle(t,x) - \int_0^t \int_{\mathrm{Y}} \sigma(x,y) e^{-(t-s)\mathcal{L}_\sigma} \mathcal{L}_1 f(s,x,y)\,\mathrm{d}y\,\mathrm{d}s$$

$$- \int_{\mathrm{Y}} \sigma(x,y) e^{-t\mathcal{L}_\sigma} \mathcal{L}_1 u_{\rm in}(x,y)\,\mathrm{d}y.$$

Explicit solution:

$$u^\epsilon(t,x) = u_{\rm in}\left(x, \frac{x}{\epsilon}\right) e^{-\sigma^\epsilon(x)t} + \int_0^t e^{-\sigma^\epsilon(x)(t-s)} f\left(s, x, \frac{x}{\epsilon}\right) \, {\rm d}s.$$

The regularity properties of the initial condition $u_{\rm in}$ and of the source term f, together with the fact that $\sigma^\epsilon \geq 0$, imply that, uniformly in ϵ

$$\|u^\epsilon\|_{{\rm L}^\infty((0,T);{\rm L}^2(\Omega))} \leq C < \infty$$

Nguetseng & Allaire's theorem guarantees the existence of a subsequence u^ϵ which two-scale converges to a function $u^0 \in {\rm L}^2((0,T) \times \Omega \times {\rm Y})$

Equation satisfied by the limit u^0

Passing to the limit as $\epsilon \to 0$ in the sense of two-scale, we obtain

$$u^0(t,x,y) = u_{\rm in}(x,y)e^{-\sigma(x,y)t} + \int_0^t e^{-\sigma(x,y)(t-s)} f(s,x,y)\, {\rm d}s$$

i.e., u^0 solves

$$\begin{cases} \partial_t u^0(t,x,y) + \sigma(x,y)u^0(t,x,y) = f(t,x,y) & (t,x,y) \in (0,T)\times\Omega\times\mathrm{Y} \\ u^0(0,x,y) = u_{\rm in}(x,y) & (x,y)\in\Omega\times\mathrm{Y} \end{cases}$$

Decomposition

u^ϵ converges weakly in $\mathrm{L}^2((0,T)\times\Omega)$ to

$$u_{\mathrm{hom}}(t,x) = \langle u^0\rangle(t,x)$$

and we can then decompose the two-scale limit into a homogeneous part and a remainder which is of zero mean over the periodic cell:

$$u^0(t,x,y) = u_{\mathrm{hom}}(t,x) + r(t,x,y) \qquad \text{where} \quad \langle r\rangle = 0.$$

We have

$$\partial_t u_{\mathrm{hom}} + \sigma(x,y)u_{\mathrm{hom}} + \partial_t r + \sigma(x,y) r = f(t,x,y).$$

Integrating the above equation over the periodicity cell Y yields

$$\partial_t u_{\mathrm{hom}} + \langle\sigma\rangle(x)u_{\mathrm{hom}} = \langle f\rangle(t,x) - \langle \sigma(x,\cdot) r(t,x,\cdot)\rangle$$

as the reminder r is of zero average in the y variable.

The coupled system for $u_{\text{hom}}(t,x)$ and $r(t,x,y)$

Equation for the remainder term:

$$\partial_t r + \sigma(x,y) r - \int_{\mathrm{Y}} \sigma(x,y) r(t,x,y)\,\mathrm{d}y$$

$$= \Big(\langle \sigma \rangle (x) - \sigma(x,y) \Big) u_{\text{hom}} + f(t,x,y) - \langle f \rangle (t,x).$$

Coupled system for $u_{\text{hom}}(t,x)$ and $r(t,x,y)$

$$\begin{cases} \partial_t u_{\text{hom}} + \langle \sigma \rangle (x) u_{\text{hom}} = \langle f \rangle (t,x) - \langle \sigma(x,\cdot) r(t,x,\cdot) \rangle \\ \partial_t r + \mathcal{L}_\sigma r = -u_{\text{hom}} \mathcal{L}_1 \sigma + \mathcal{L}_1 f \\ u_{\text{hom}}(0,x) = \langle u_{\text{in}}(x) \rangle \\ r(0,x,y) = \mathcal{L}_1 u_{\text{in}}. \end{cases}$$

The decoupled equation for $u_{\mathrm{hom}}(t, x)$

Solve for the remainder term $r(t, x, y)$ in terms of u_{hom}

$$r(t,x,y) = e^{-t\mathcal{L}_\sigma}\mathcal{L}_1 u_{\mathrm{in}}(x,y) + \int_0^t e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 f(s,x,y)\,\mathrm{d}s$$

$$- \int_0^t e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 \sigma(x,y) u_{\mathrm{hom}}(s,x)\,\mathrm{d}s$$

Substitute this expression for the remainder in the evolution for u_{hom}

$$\partial_t u_{\mathrm{hom}} + \langle\sigma\rangle(x) u_{\mathrm{hom}} = \langle f\rangle(t,x)$$

$$+ \int_0^t \int_{\mathrm{Y}} \sigma(x,y) e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 \sigma(x,y) u_{\mathrm{hom}}(s,x)\,\mathrm{d}y\,\mathrm{d}s$$

$$- \int_0^t \int_{\mathrm{Y}} \sigma(x,y) e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 f(s,x,y)\,\mathrm{d}y\,\mathrm{d}s$$

$$- \int_{\mathrm{Y}} \sigma(x,y) e^{-t\mathcal{L}_\sigma}\mathcal{L}_1 u_{\mathrm{in}}(x,y)\,\mathrm{d}y$$

Extension

Assume that $\mathbf{\Sigma}^\epsilon$ is diagonalizable in the sense that there exists $\mathbf{P} \in \mathbb{R}^{n\times n}$ invertible and $\mathbf{D}^\epsilon \in \mathbb{R}^{n\times n}$ diagonal such that $\mathbf{\Sigma}^\epsilon = \mathbf{P}\mathbf{D}^\epsilon\mathbf{P}^{-1}$.

$$\begin{cases} \partial_t \mathbf{u}^\epsilon(t,x) + \mathbf{\Sigma}^\epsilon(x)\mathbf{u}^\epsilon(t,x) = \mathbf{f}^\epsilon(t,x), \qquad \mathbf{u}^\epsilon \in \mathbb{R}^n \\ \mathbf{u}^\epsilon(0,x) = \mathbf{u}_{\rm in}\left(x, \frac{x}{\epsilon}\right) \end{cases}$$

$$\mathbf{\Sigma}^\epsilon(x) = \mathbf{\Sigma}\left(x, \frac{x}{\epsilon}\right) = \left(\Sigma_{i,j}\left(x, \frac{x}{\epsilon}\right)\right)_{1\le i,j\le n} \qquad \Sigma_{i,j} \in \mathrm{L}^\infty(\Omega; \mathrm{C}_{\rm per}(\mathrm{Y}))$$

$$\mathbf{f}^\epsilon(t,x) = \mathbf{f}\left(t, x, \frac{x}{\epsilon}\right) = \left(f_i\left(t, x, \frac{x}{\epsilon}\right)\right)_{1\le i\le n} \qquad f_i^\epsilon \in \mathrm{L}^\infty((0,T)\times\Omega; \mathrm{C}_{\rm per}(\mathrm{Y}))$$

Equivalence with Tartar's formulation

Suppose $\sigma(x, y) = \sigma(y)$ (purely periodic absorption term)

The memory kernel $\mathcal{K}$ takes the form

$$\mathcal{K}(\tau) := \int_{\mathrm{Y}} \sigma(y) e^{-\tau \mathcal{L}_\sigma} \left(\sigma - \langle \sigma \rangle\right)(y) \, \mathrm{d}y$$

Proposition (H. Hutridurga, O. Mula, FS)

For any $p > 0$,

$$\widehat{\mathcal{K}}(p) = \widehat{\mathcal{M}}(p) = p + \langle \sigma \rangle - \mathcal{B}(p) = \int_{\mathrm{Y}} \Big(p + \sigma(y) - \mathcal{B}(p)\Big) \, \mathrm{d}y.$$

Proof

The Laplace transform for the memory kernel is

$$\widehat{\mathcal{K}}(p) = \int_0^\infty e^{-pt}\mathcal{K}(t)\,\mathrm{d}t = \int_0^\infty \int_{\mathrm{Y}} \sigma(y) e^{-pt} e^{-t\mathcal{L}_\sigma}\left(\sigma - \langle\sigma\rangle\right)(y)\,\mathrm{d}y\,\mathrm{d}t$$

The Laplace transform of a semigroup yields the corresponding resolvent:

$$\widehat{\mathcal{K}}(p) = \int_{\mathrm{Y}} \sigma(y)\left[p + \mathcal{L}_\sigma\right]^{-1}\left(\sigma - \langle\sigma\rangle\right)(y)\,\mathrm{d}y.$$

Consider now the equation

$$\left[p + \mathcal{L}_\sigma\right] g(y) = f(y), \quad y \in \mathrm{Y}$$

for a given $p > 0$ and a given measurable function f of zero mean.

Averaging the equation in the y variable yields that it is necessary to have the solution $g(y)$ to be of zero average as well. Hence, for zero average functions:

$$[p + \mathcal{L}_\sigma]\, g(y) = p + \sigma(y)g(y) - \int_{\mathrm{Y}} \sigma(y)g(y)\,\mathrm{d}y$$

$$= p + \sigma(y)g(y) - \int_{\mathrm{Y}} (\sigma(y) + p)\, g(y)\,\mathrm{d}y = \mathcal{L}_{p+\sigma} g(y)$$

A simple inspection reveals that a general solution to $\mathcal{L}_{p+\sigma} g(y) = f(y)$ is given by

$$g(y) = \frac{f(y)}{p + \sigma(y)} + \frac{C}{p + \sigma(y)}$$

where C needs to be chosen such that $g(y)$ is of zero average:

$$C = -\mathcal{B}(p) \int_{\mathrm{Y}} \frac{f(y)}{p + \sigma(y)}\,\mathrm{d}y \qquad \text{with} \qquad \mathcal{B}(p) := \left(\int_{\mathrm{Y}} \frac{\mathrm{d}y}{p + \sigma(y)}\right)^{-1}$$

As a result, by taking $f(y) = \sigma - \langle\sigma\rangle$:

$$[p + \mathcal{L}_\sigma]^{-1}(\sigma - \langle\sigma\rangle)(y) = g(y) = \frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma} - \frac{\mathcal{B}(p)}{p+\sigma}\int_{\mathrm{Y}} \frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma(y)}\,\mathrm{d}y$$

Using the above observation, we have:

$$\begin{aligned}\widehat{\mathcal{K}}(p) &= \int_{\mathrm{Y}} \sigma(y)\left(\frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma} - \frac{\mathcal{B}(p)}{p+\sigma}\int_{\mathrm{Y}} \frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma(y)}\,\mathrm{d}y\right)\mathrm{d}y \\ &= \int_{\mathrm{Y}} (\sigma(y)+p)\left(\frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma} - \frac{\mathcal{B}(p)}{p+\sigma}\int_{\mathrm{Y}} \frac{\sigma(y) - \langle\sigma\rangle}{p+\sigma(y)}\,\mathrm{d}y\right)\mathrm{d}y\end{aligned}$$

as $g(y)$ is of zero mean. Some further computations show that

$$\widehat{\mathcal{K}}(p) = \langle\sigma\rangle - \mathcal{B}(p)\int_{\mathrm{Y}} \frac{\sigma(y)}{p+\sigma(y)}\,\mathrm{d}y = p + \langle\sigma\rangle - \mathcal{B}(p) = \widehat{\mathcal{M}}(p),$$

thus proving the equivalence.

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The rapidly oscillating problem for the linear Boltzmann equation ($0 < \varepsilon \ll 1$)

$$\sqrt{\frac{m}{2E}}\,\partial_t \varphi^\varepsilon + \omega \cdot \nabla_x \varphi^\varepsilon + \sigma^\varepsilon (x, \omega, E)\, \varphi^\varepsilon$$

$$- \int_{E_{\min}}^{E_{\max}} \int_{|\omega'|=1} \kappa^\varepsilon \left(x, \omega \cdot \omega', E, E'\right) \varphi^\varepsilon(x, \omega', E') \mathrm{d}\omega' \, \mathrm{d}E' = 0$$

$$\sigma^\varepsilon(x, \omega, E) = \sigma\left(x, \omega, E, \frac{E}{\varepsilon}\right)$$

$$\kappa^\varepsilon(x, \omega \cdot \omega', E, E') = \kappa\left(x, \omega \cdot \omega', E, E', \frac{E'}{\varepsilon}\right)$$

$\sigma(x, \omega, E, y)$ and $\kappa\,(x, \omega \cdot \omega', E, E', y')$ are assumed to be periodic in the y and y' variables respectively.

The equation is complemented with zero incoming flux condition on the boundary and initial condition $\varphi_{\text{in}} \in \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1} \times (E_{\min}, E_{\max}))$

Further hypotheses on the optical parameters

Let

$$\bar{\kappa}^\varepsilon(x,\omega,E) = \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \kappa^\varepsilon(x,\omega\cdot\omega',E,E')\,\mathrm{d}\omega'\,\mathrm{dE}'$$

$$\tilde{\kappa}^\varepsilon(x,\omega,E) = \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \kappa^\varepsilon(x,\omega\cdot\omega',E',E)\,\mathrm{d}\omega'\,\mathrm{dE}'.$$

Assume that there exists $\alpha > 0$ such that for all $\varepsilon > 0$,

$$\sigma^\varepsilon(x,\omega,E) - \bar{\kappa}^\varepsilon(x,\omega,E) \geq \alpha \quad \text{and} \quad \sigma^\varepsilon(x,\omega,E) - \tilde{\kappa}^\varepsilon(x,\omega,E) \geq \alpha$$

Hypothesis on the kernel structure

κ^ε exhibits separation in the E and E' variables:

$$\kappa^\varepsilon(x,\omega\cdot\omega',E,E') := \kappa_1(x,\omega\cdot\omega',E)\kappa_2\left(x,\omega\cdot\omega',E',\frac{E'}{\varepsilon}\right)$$

with $\kappa_2\left(x,\omega\cdot\omega',E',y'\right)$ being periodic in the y' variable.

Positivity property of the Boltzmann operator

Here $\langle\cdot\rangle$ denotes integral over the interval $(0,1)$, i.e. averaging over the periodic cell in the energy variable:

$$\langle v\rangle := \int_0^1 v(y)\,\mathrm{d}y \qquad \text{for all } v \in \mathrm{L}^1(0,1).$$

Notation: $\mathbb{V} = \mathbb{S}^{d-1} \times (E_{\min}, E_{\max})$

$$\mathcal{Q}^\varepsilon f := \sigma^\varepsilon f - \int_{\mathbb{V}} \kappa^\varepsilon(x,\omega\cdot\omega',E,E')f(x,\omega',E')\,\mathrm{d}\omega'\,\mathrm{d}E', \quad \forall f \in \mathrm{L}^2(\Omega\times\mathbb{V}).$$

Proposition

If $(\sigma^\varepsilon,\kappa^\varepsilon)$ satisfy the previous assumptions, then for all $\varepsilon>0$ and all $f\in\mathrm{L}^2(\Omega\times\mathbb{V})$,

$$(\mathcal{Q}^\varepsilon f, f)_{\mathrm{L}^2(\Omega\times\mathbb{V})} \geq \alpha\|f\|^2_{\mathrm{L}^2(\Omega\times\mathbb{V})}.$$

Proof

From the Cauchy-Schwarz inequality and the definition of $\bar\kappa^\varepsilon$ and $\tilde\kappa^\varepsilon$:

$$\left(\int_{\mathbb{V}}\int_{\mathbb{V}} f(x,\omega,E) f(x,\omega',E')\kappa^\varepsilon(x,\omega\cdot\omega',E,E')\,\mathrm{d}\omega\,\mathrm{dE}\,\mathrm{d}\omega'\,\mathrm{dE}'\right)^2$$

$$\leq \left(\int_{\mathbb{V}}\int_{\mathbb{V}} |f(x,\omega,E)|^2\,\kappa^\varepsilon(x,\omega\cdot\omega',E,E')\,\mathrm{d}\omega\,\mathrm{dE}\,\mathrm{d}\omega'\,\mathrm{dE}'\right)$$

$$\left(\int_{\mathbb{V}}\int_{\mathbb{V}} \left|f(x,\omega',E')\right|^2\,\kappa^\varepsilon(x,\omega\cdot\omega',E,E')\,\mathrm{d}\omega\,\mathrm{dE}\,\mathrm{d}\omega'\,\mathrm{dE}'\right)$$

$$= \left(\int_{\mathbb{V}} |f(x,\omega,E)|^2\,\bar\kappa^\varepsilon(x,\omega,E)\,\mathrm{d}\omega\,\mathrm{dE}\right)\left(\int_{\mathbb{V}} |f(x,\omega,E)|^2\,\tilde\kappa^\varepsilon(x,\omega,E)\,\mathrm{d}\omega\,\mathrm{dE}\right)$$

Thanks to Young's inequality and our assumptions:

$$(\mathcal{Q}^\varepsilon f, f)_{\mathrm{L}^2(\Omega\times\mathbb{V})} \geq \int_\Omega \mathrm{d}x \int_{\mathbb{V}} |f(x,\omega,E)|^2\, \sigma^\varepsilon(x,\omega,E)\,\mathrm{d}\omega\,\mathrm{dE}$$

$$-\int_\Omega \left(\int_{\mathbb{V}} |f(x,\omega,E)|^2\, \bar{\kappa}^\varepsilon(x,\omega,E)\,\mathrm{d}\omega\,\mathrm{dE} \right)^{1/2} \left(\int_{\mathbb{V}} |f(x,\omega,E)|^2\, \tilde{\kappa}^\varepsilon(x,\omega,E)\,\mathrm{d}\omega\,\mathrm{dE} \right)$$

$$\geq \alpha \|f\|^2_{\mathrm{L}^2(\Omega\times\mathbb{V})}$$

Lemma

If $(\sigma^\varepsilon, \kappa^\varepsilon)$ satisfy the previous assumptions, then there exists $C > 0$ such that for all $\varepsilon > 0$, the solution φ^ε satisfies

$$\|\varphi^\varepsilon\|_{\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega\times\mathbb{V}))} \leq C \quad \text{and} \quad \|\varphi^\varepsilon\|_{\mathrm{L}^2((0,T)\times\Omega\times\mathbb{V})} \leq C.$$

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The main theorem

Theorem (H. Hutridurga, O. Mula, FS)

Let $\varphi^\varepsilon = \varphi^\varepsilon(t, x, \omega, E)$ be the solution of the equation

$$\partial_t \varphi^\varepsilon + \sqrt{E}\,\omega \cdot \nabla_x \varphi^\varepsilon + \sigma^\varepsilon\,(\omega, E)\,\varphi^\varepsilon -$$

$$\int_{E_{\min}}^{E_{\max}} \int_{|\omega'|=1} \kappa^\varepsilon\,(\omega \cdot \omega', E, E')\,\varphi^\varepsilon(\omega', E')\mathrm{d}\omega'\,\mathrm{d}E' = 0$$

$$\varphi^\varepsilon(0, x, \omega, E) = \varphi^\varepsilon_{\mathrm{in}}\,(x, \omega, E)$$

$$\varphi^\epsilon(t, x, \omega, E) = 0 \quad \forall t > 0 \quad \text{and for } (x, \omega) \in \Gamma_-$$

where the coefficients and the data are of the form ...

$$\sigma^\varepsilon(\omega, E) := \sqrt{E}\,\sigma\left(\omega, E, \frac{E}{\epsilon}\right) \text{ with } \sigma \in \mathrm{L}^\infty(\mathbb{V}; \mathrm{C}_{\mathrm{per}}(0,1))$$

$$\varphi_{\mathrm{in}}^\varepsilon(x, \omega, E) := \varphi_{\mathrm{in}}\left(x, \omega, E, \frac{E}{\epsilon}\right) \text{ with } \varphi_{\mathrm{in}} \in \mathrm{L}^2(\Omega \times \mathbb{V}; \mathrm{C}_{\mathrm{per}}(0,1))$$

$$\kappa^\varepsilon(\omega \cdot \omega', E, E') := \sqrt{E}\,\kappa_1(\omega \cdot \omega', E)\kappa_2\left(\omega \cdot \omega', E', \frac{E'}{\varepsilon}\right) \text{ with}$$

$$\kappa_1 \in \mathrm{L}^\infty\left([-1,1] \times [E_{\mathrm{min}}, E_{\mathrm{max}}]\right) \text{ and}$$

$$\kappa_2 \in \mathrm{L}^\infty\left([-1,1] \times [E_{\mathrm{min}}, E_{\mathrm{max}}]; \mathrm{C}_{\mathrm{per}}(0,1)\right).$$

Then,

$$\varphi^\epsilon \rightharpoonup \varphi_{\mathrm{hom}} \quad \text{weakly in } \mathrm{L}^2((0,T) \times \Omega \times \mathbb{V})$$

and φ_{hom} satisfies the partial integro-differential equation . . .

$$\partial_t \varphi_{\mathrm{hom}} + \sqrt{E}\,\omega\cdot\nabla_x\varphi_{\mathrm{hom}} + \sqrt{E}\,\langle\sigma\rangle\varphi_{\mathrm{hom}}$$

$$-\int_{E_{\mathrm{min}}}^{E_{\mathrm{max}}}\int_{\mathbb{S}^{d-1}}\sqrt{E}\,\kappa_1(\omega\cdot\omega',E)\int_0^1\kappa_2(\omega\cdot\omega',E',y')\varphi_{\mathrm{hom}}\mathrm{d}y'\mathrm{d}\omega'\mathrm{d}E' =$$

$$\int_{E_{\mathrm{min}}}^{E_{\mathrm{max}}}\int_{\mathbb{S}^{d-1}}\sqrt{E}\,\kappa_1(\omega\cdot\omega',E)\int_0^1\kappa_2(\omega\cdot\omega',E',y')\times$$

$$\left[e^{-t\sqrt{E'}\mathcal{L}_\sigma}\mathcal{L}_1\varphi_{\mathrm{in}} - \int_0^t e^{-(t-s)\sqrt{E'}\mathcal{L}_\sigma}\sqrt{E'}\mathcal{L}_1\sigma(\omega',E',y')\varphi_{\mathrm{hom}}\,\mathrm{d}s\right]\mathrm{d}y'\mathrm{d}\omega'\mathrm{d}E'$$

$$-\sqrt{E}\int_0^1\sigma(\omega,E,y)\left[e^{-t\sqrt{E}\mathcal{L}_\sigma}\mathcal{L}_1\varphi_{\mathrm{in}} - \int_0^t e^{-(t-s)\sqrt{E}\mathcal{L}_\sigma}\sqrt{E}\mathcal{L}_1\sigma(\omega,E,y)\times\right.$$

$$\left.\varphi_{\mathrm{hom}}\,\mathrm{d}s\right]\mathrm{d}y,$$

with initial condition $\varphi_{\mathrm{hom}}(0,x,\omega,E) = \langle\varphi_{\mathrm{in}}(x,\omega,E,\cdot)\rangle$ and zero absorption condition at the in-flux phase-space boundary.

Concluding remarks on the assumption on the optical parameters σ^ε and κ^ε

- The assumption of separability

$$\kappa^\varepsilon(\omega \cdot \omega', E, E') := \sqrt{E}\, \kappa_1(\omega \cdot \omega', E)\kappa_2\left(\omega \cdot \omega', E', \frac{E'}{\varepsilon}\right)$$

  simplifies the computations in the proof. It also lead to a relatively simpler homogenized model.
- In the above separable structure, we can further allow the factor κ_1 to oscillate in the E-variable. The proof of the main theorem can be reworked in this case, at the price of arriving at a more complex two-scale system. The memory structure remains the same but with additional terms.
- It is apparent from the proof of the main theorem that the energy oscillations in σ, not those in the scattering kernel, resulted in the memory effects in the homogenized limit.

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Numerical illustration of the homogenization limit

$$\begin{cases} \partial_t \varphi^\epsilon(t,E) + \sigma\left(\frac{E}{\epsilon}\right)\varphi^\epsilon(t,E) = \int_{E_{\min}}^{E_{\max}} \kappa\left(\frac{E'}{\epsilon}\right)\varphi^\epsilon(t,E')\,\mathrm{d}E' \\ \varphi^\epsilon(0,E) = \varphi_{\mathrm{in}}(E) \end{cases}$$

for $(t,E) \in [0,10] \times (E_{\min}, E_{\max}) = (0,1)$.

Strategy: family of orthogonal Legendre polynomials in $\mathrm{L}^2(E_{\min}, E_{\max})$ denoted by $\{\ell_k\}_{k \ge 0}$

Define the modes

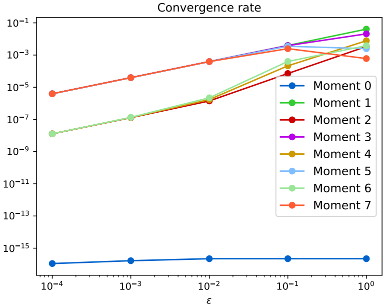
$$m_k^\epsilon(t) = (\varphi^\epsilon(t,\cdot), \ell_k(\cdot))_{\mathrm{L}^2([E_{\min},E_{\max}])}\,, \quad k \ge 0$$

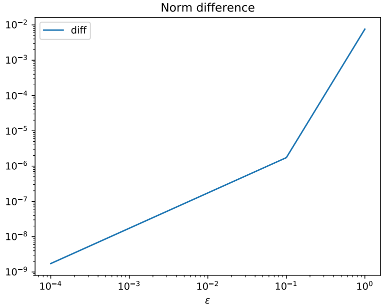
of the solution for $t \in [0,T]$. Likewise,

$$m_k^{\mathrm{hom}}(t) := (\varphi_{\mathrm{hom}}(t,\cdot), \ell_k(\cdot))_{\mathrm{L}^2([E_{\min},E_{\max}])}\,, \quad k \ge 0$$

are the modes of the homogenized solution φ_{hom}.

Numerical simulations for $\sigma(y) = 2 + \frac{1}{2}\sin(2\pi y)$, $\kappa(y') = 1 + \frac{1}{2}\sin(2\pi y')$, $\varphi_{\rm in}(y) = 1 + \sin(2\pi y)$





Convergence rate in ϵ of the error $e_k^\epsilon = \max_{t\in[0,T]} |m_k^\epsilon(t) - m_k^{\rm hom}(t)|$ (left) and norm difference $|\|\varphi^\epsilon\|_{\rm L^2} - \|\varphi^0\|_{\rm L^2}|$ (right)

THANK YOU FOR YOUR ATTENTION