Large-magnetic field regimes & asymptotic preserving schemes

Miguel Rodrigues

IRMAR,
Univ Rennes &
Institut Universitaire de France

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Jointly with Francis Filbet (Toulouse 3, I.U.F.).
Parts also with Hamed Zakerzadeh (Toulouse 3).

Context.

In **plasma** dynamics forced by **strong magnetic fields**, coexist

- strong oscillations ensuring confinement of energetic particles;
- a slow dynamics uncoupled at leading order.

Goal: prove the above claim and design suitable numerical schemes.

This is the so-called **gyrokinetic** theory.

State of the art of the analytic theory (even now):

- either geometrically trivial PDEs; (St-Raymond, Miot, Bostan)
- or linear PDEs. (Frénod-Lutz, Possaner)

Outline.

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1. Elements of gyrokinetics
   - Mechanism
   - Uniform magnetic fields
   - PDE counterparts
   - Two-dimensional fields
   - Three-dimensional fields
2. A few asymptotic preserving shemes
3. Conclusion

A dumb example.

$(\mathbf{x}^\varepsilon, \mathbf{v}^\varepsilon) : \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}^2$.

$$\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) = \mathbf{v}^\varepsilon(t), \qquad \frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp .$$

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Short-time uniform bounds

$$\|\mathbf{v}^\varepsilon(t)\| = \|\mathbf{v}^\varepsilon(0)\|, \qquad \|\mathbf{x}^\varepsilon(t)\| \leq \|\mathbf{x}^\varepsilon(0)\| + |t|\,\|\mathbf{v}^\varepsilon(0)\|.$$

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Uncoupling $\mathbf{x}^\varepsilon$ from $\mathbf{v}^\varepsilon$

$$\mathbf{v}^\varepsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon(\mathbf{v}^\varepsilon)^\perp\right)(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}^\varepsilon - \varepsilon(\mathbf{v}^\varepsilon)^\perp\right)(t) = 0.$$

Thus

$$\|\mathbf{x}^\varepsilon(t) - \mathbf{x}^\varepsilon(0)\| \leq 2\,\varepsilon\,\|\mathbf{v}^\varepsilon(0)\|.$$

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$$\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) = \mathbf{v}^\varepsilon(t), \qquad \frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp.$$

Improved uniform bounds

$$\|\mathbf{v}^\varepsilon(t)\| = \|\mathbf{v}^\varepsilon(0)\|, \qquad \|\mathbf{x}^\varepsilon(t)\| \le \|\mathbf{x}^\varepsilon(0)\| + \min(\{|t|, 2\varepsilon\})\,\|\mathbf{v}^\varepsilon(0)\|.$$

Uncoupling $\mathbf{x}^\varepsilon$ from $\mathbf{v}^\varepsilon$

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Outline.

Uniform magnetic fields, transverse dynamics.

$$\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) = \mathbf{v}^\varepsilon(t) \qquad \frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp + \mathbf{E}^\varepsilon(t,\mathbf{x}^\varepsilon(t))\,.$$

Uniform magnetic fields, transverse dynamics.

$$\frac{d\mathbf{x}^\varepsilon}{dt}(t) = \mathbf{v}^\varepsilon(t) \qquad \frac{d\mathbf{v}^\varepsilon}{dt}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp + \mathbf{E}^\varepsilon(t,\mathbf{x}^\varepsilon(t))\,.$$

Short-time uniform bounds.

$$\|\mathbf{v}^\varepsilon(t)\| \leq \|\mathbf{v}^\varepsilon(0)\| + 2\,|t|\,\|\mathbf{E}^\varepsilon\|_{L^\infty}\,,$$
$$\|\mathbf{x}^\varepsilon(t)\| \leq \|\mathbf{x}^\varepsilon(0)\| + |t|\,\|\mathbf{v}^\varepsilon(0)\| + t^2\,\|\mathbf{E}^\varepsilon\|_{L^\infty}\,.$$

$$\mathbf{v}^\varepsilon(t) = \frac{d}{dt}\left(\varepsilon(\mathbf{v}^\varepsilon)^\perp\right)(t) - \varepsilon(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t))\,,$$
$$\frac{d}{dt}\left(\mathbf{x}^\varepsilon - \varepsilon(\mathbf{v}^\varepsilon)^\perp\right)(t) = -\varepsilon(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t))\,.$$

Thus

$$\|\mathbf{x}^\varepsilon(t) - \mathbf{x}^\varepsilon(0)\| \leq \varepsilon\,\left(2\,\|\mathbf{v}^\varepsilon(0)\| + 3\,|t|\,\|\mathbf{E}^\varepsilon\|_{L^\infty}\right)\,.$$

Uniform magnetic fields, second order.

Guiding center variable

$$\mathbf{x}^{\varepsilon}_{\mathrm{gc}} := \mathbf{x}^{\varepsilon} - \varepsilon(\mathbf{v}^{\varepsilon})^{\perp}$$

so that

$$\frac{\mathrm{d}\mathbf{x}^{\varepsilon}_{\mathrm{gc}}}{\mathrm{d}t}(t) = -\varepsilon\,(\mathbf{E}^{\varepsilon})^{\perp}(t, \mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t) + \varepsilon\,(\mathbf{v}^{\varepsilon})^{\perp})\,.$$

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Proposition

$$\left\|\mathbf{x}_{\mathrm{gc}}^{\varepsilon}(t) - \mathbf{y}_{\mathrm{gc}}^{\varepsilon}(t)\right\| \leq \varepsilon^2 \|\mathrm{d}_{\mathbf{x}}\mathbf{E}\|_{L^\infty}\, \mathrm{e}^{\varepsilon\,|t|\,\|\mathrm{d}_{\mathbf{x}}\mathbf{E}\|_{L^\infty}} \left(\, t\,\|\mathbf{v}^{\varepsilon}(0)\| + t^2\,\|\mathbf{E}\|_{L^\infty}\,\right)$$

where $\mathbf{y}_{\mathrm{gc}}^{\varepsilon}$ solves

$$\frac{\mathrm{d}\mathbf{y}_{\mathrm{gc}}^{\varepsilon}}{\mathrm{d}t}(t) = -\varepsilon\,(\mathbf{E}^{\varepsilon})^{\perp}(t, \mathbf{y}_{\mathrm{gc}}^{\varepsilon}(t))\,,$$
$$\mathbf{y}_{\mathrm{gc}}^{\varepsilon}(0) = \mathbf{x}_{\mathrm{gc}}^{\varepsilon}(0)\,.$$

Large-time uniform estimates, first way.

$$\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) = \mathbf{v}^\varepsilon(t) \qquad \frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp + \mathbf{E}^\varepsilon(t, \mathbf{x}^\varepsilon(t)).$$

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Introducing a more purely oscillatory variable.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}^\varepsilon + \varepsilon(\mathbf{E}^\varepsilon)^\perp(\cdot,\mathbf{x}^\varepsilon)\right)(t) = -\frac{1}{\varepsilon}\left(\mathbf{v}^\varepsilon(t) + \varepsilon(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t))\right)^\perp$$
$$+\,\varepsilon\left(\partial_t(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t)) + \mathrm{d}_\mathbf{x}(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t))(\mathbf{v}^\varepsilon(t))\right).$$

Thus

$$\|\mathbf{v}^\varepsilon(t)\| \leq e^{\varepsilon\,|t|\,\|\mathrm{d}_\mathbf{x}\mathbf{E}^\varepsilon\|_{L^\infty}}\Big(\|\mathbf{v}^\varepsilon(0)\| + 2\,\varepsilon\,\|\mathbf{E}^\varepsilon\|_{L^\infty}$$
$$+\,\varepsilon\,|t|\,\left(\|\partial_t\mathbf{E}^\varepsilon\|_{L^\infty} + \|\mathrm{d}_\mathbf{x}\mathbf{E}^\varepsilon\|_{L^\infty}\|\mathbf{E}^\varepsilon\|_{L^\infty}\right)\Big)$$

Large-time uniform estimates, second way.

Kinetic energy variable

$$e(\mathbf{v}^\varepsilon) := \frac{1}{2}\|\mathbf{v}^\varepsilon\|^2$$

as a slow variable.

$$\frac{\mathrm{d}e(\mathbf{v}^\varepsilon)}{\mathrm{d}t}(t) = \langle \mathbf{v}^\varepsilon, \mathbf{E}^\varepsilon(t, \mathbf{x}^\varepsilon(t)) \rangle .$$

Large-time uniform estimates, second way.

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Introducing a slower variable, uncoupling e^ε from $\mathbf{v}^\varepsilon$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(e(\mathbf{v}^\varepsilon) - \varepsilon\langle(\mathbf{v}^\varepsilon)^\perp, \mathbf{E}^\varepsilon(\cdot, \mathbf{x}^\varepsilon)\rangle\right)(t)\\ &= -\varepsilon\left\langle(\mathbf{v}^\varepsilon)^\perp, (\partial_t\mathbf{E}^\varepsilon(t, \mathbf{x}^\varepsilon(t)) + \mathrm{d}_{\mathbf{x}}\mathbf{E}^\varepsilon(t, \mathbf{x}^\varepsilon(t))(\mathbf{v}^\varepsilon(t)))\right\rangle\,.\end{aligned}$$

Same kind of conclusion.

Uniform magnetic fields, large time.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}^\varepsilon - \varepsilon(\mathbf{v}^\varepsilon)^\perp\right)(t) = -\varepsilon(\mathbf{E}^\varepsilon)^\perp(t, \mathbf{x}^\varepsilon(t))\,.$$

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Proposition

$$\|\mathbf{x}^\varepsilon(t) - \mathbf{y}^\varepsilon(t)\| \leq \varepsilon\, \mathrm{e}^{2\,\varepsilon\,|t|\|\mathrm{d}_\mathbf{x}\mathbf{E}^\varepsilon\|_{L^\infty}} \Big(2\,(\|\mathbf{v}^\varepsilon(0)\| + \varepsilon\,\|\mathbf{E}^\varepsilon\|_{L^\infty}) + \varepsilon\,|t|\,(\|\partial_t\mathbf{E}^\varepsilon\|_{L^\infty} + \|\mathrm{d}_\mathbf{x}\mathbf{E}^\varepsilon\|_{L^\infty}\|\mathbf{E}^\varepsilon\|_{L^\infty})\Big)$$

where $\mathbf{y}^\varepsilon$ solves

$$\frac{\mathrm{d}\mathbf{y}^\varepsilon}{\mathrm{d}t}(t) = -\varepsilon\,(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{y}^\varepsilon(t))\,,$$
$$\mathbf{y}^\varepsilon(0) = \mathbf{x}^\varepsilon(0)\,.$$

Outline.

From nonlinear ODEs...

Let Φ and $\Phi_{\rm slow}$ be flows associated with respective ODEs

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathcal{X}(t,\mathbf{X}(t)) \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) = \mathcal{X}_{\rm slow}(t,\mathbf{a}(t))$$

and $\mathcal{A}$ and $\mathcal{M}$ be slow maps and weights such that, for a.e. $t \geq 0$,

$$\|\mathcal{A}(t,\Phi(t,0,\mathbf{X}_0)) - \Phi_{\rm slow}(t,0,\mathcal{A}(0,\mathbf{X}_0))\| \leq \mathcal{M}(t,\mathbf{X}_0)\,.$$

...to linear PDEs.

Then if f solves

$$\partial_t f + \mathrm{div}_{\mathbf{x}}(\mathcal{X} f) = 0, \qquad f(0,\cdot) = f_0\,,$$

and $F(t,\cdot) = \mathcal{A}(t,\cdot)_* f(t,\cdot)$ is the push-forward of f by the slow map $\mathcal{A}$ then for a.e. $t \geq 0$

$$\|F(t,\cdot) - G(t,\cdot)\|_{\dot{W}^{-1,1}} \leq \int \mathcal{M}(t,\cdot)\mathrm{d}|f_0|$$

where G solves

$$\partial_t G + \mathrm{div}_{\mathbf{a}}(\mathcal{X}_{\mathrm{as}}\, G) = 0, \qquad G(0,\cdot) = \mathcal{A}(0,\cdot)_* f_0\,.$$

Pushing forward and averaging.

Definition: dual to composition, $\langle \mathcal{A}_* \mu, \varphi \rangle := \langle \mu, \varphi \circ \mathcal{A} \rangle$.

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Pushing by **characteristic flow** (from 0 to t):

$$\partial_t f + \mathrm{div}_{\mathbf{x}}(\mathcal{X} f) = 0 \qquad \text{solved by} \qquad f(t,\cdot) = \Phi(t,0,\cdot)_*(f(0,\cdot))\,.$$

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In regular cases, **averaging** over level sets

$$\mathcal{A}_*(f)(a) = \int_{\mathcal{A}^{-1}(\{a\})} f(\mathbf{X}) \frac{\mathrm{d}\sigma_a(\mathbf{X})}{\sqrt{\det(\mathrm{d}\mathcal{A}(\mathbf{X})(\mathrm{d}\mathcal{A}(\mathbf{X}))^*)}}$$

where σ_a denotes the surface measure on $\mathcal{A}^{-1}(\{a\})$.

The Vlasov equation.

$$\partial_t f^\varepsilon + \mathrm{div}_{\mathbf{x}}(f^\varepsilon \mathbf{v}) + \mathrm{div}_{\mathbf{v}}\left(f^\varepsilon\left(\mathbf{E}^\varepsilon(t,\mathbf{x}) - \frac{1}{\varepsilon}\mathbf{v}^\perp\right)\right) = 0$$

Macroscopic density

$$\rho^\varepsilon(t,\cdot) := \mathcal{A}(t,\cdot)_* \, f^\varepsilon(t,\cdot) = \int_{\mathbf{R}^2} f^\varepsilon(t,\cdot,\mathbf{v})\,\mathrm{d}\mathbf{v}\,.$$

with $\mathcal{A}(t,(\mathbf{x},\mathbf{v})) = \mathbf{x}$.

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Corollary

$$\|\rho^\varepsilon(t,\cdot) - F^\varepsilon(t,\cdot)\|_{\dot{W}^{-1,1}} \leq \varepsilon\, C_{\varepsilon,\|E^\varepsilon\|_{W^{1,\infty}},\varepsilon t} \int (1+\|\mathbf{v}\|)\mathrm{d}|f_0|(\mathbf{x},\mathbf{v})$$

where F^ε solves

$$\frac{\partial F^\varepsilon}{\partial t} - \varepsilon\,\mathrm{div}_{\mathbf{x}}\left(F^\varepsilon(\mathbf{E}^\varepsilon)^\perp\right) = 0\,, \qquad F^\varepsilon(0,\cdot) = \rho^\varepsilon(0,\cdot)\,.$$

Outline.

Inhomogeneous magnetic field.

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) &= \mathbf{v}^\varepsilon(t)\\
\frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) &= -\frac{B^\varepsilon(t,\mathbf{x}^\varepsilon(t))}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp + \mathbf{E}^\varepsilon(t,\mathbf{x}^\varepsilon(t))
\end{aligned}
$$

with non vanishing B^ε.

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with non vanishing B^ε.

After a first elimination,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}^\varepsilon - \varepsilon\frac{(\mathbf{v}^\varepsilon)^\perp}{B^\varepsilon(\cdot,\mathbf{x}^\varepsilon)}\right)(t) \\ &\quad = -\varepsilon\frac{(\mathbf{E}^\varepsilon)^\perp(t,\mathbf{x}^\varepsilon(t))}{B^\varepsilon(t,\mathbf{x}^\varepsilon(t))} - \varepsilon(\mathbf{v}^\varepsilon)^\perp\,(\partial_t + \mathbf{v}^\varepsilon(t)\cdot\nabla_{\mathbf{x}})\left(\frac{1}{B^\varepsilon}\right)(t,\mathbf{x}^\varepsilon(t)). \end{aligned}$$

Quadratic term in $\mathbf{v}^\varepsilon$.

Back to the dumb example.

$$\frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) = -\frac{1}{\varepsilon}(\mathbf{v}^\varepsilon(t))^\perp .$$

implies for any quadratic form Q

$$\begin{aligned} Q(\mathbf{v}^\varepsilon,\mathbf{v}^\varepsilon) &= \frac{1}{2}\left(Q(\mathbf{v}^\varepsilon,\mathbf{v}^\varepsilon) + Q((\mathbf{v}^\varepsilon)^\perp,(\mathbf{v}^\varepsilon)^\perp)\right) \\ &\quad + \frac{1}{2}\left(Q(\mathbf{v}^\varepsilon,\mathbf{v}^\varepsilon) - Q((\mathbf{v}^\varepsilon)^\perp,(\mathbf{v}^\varepsilon)^\perp)\right) \\ &= \mathrm{Tr}(Q)\, e(\mathbf{v}^\varepsilon) - \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\Re(Q)(\mathbf{v}^\varepsilon,(\mathbf{v}^\varepsilon)^\perp)\right) \end{aligned}$$

where

$$e(\mathbf{v}^\varepsilon) := \frac{1}{2}\|\mathbf{v}^\varepsilon\|^2 .$$

Inhomogeneous magnetic field, large time.

Proposition

$$\left\|(\mathbf{x}^\varepsilon,\tfrac{1}{2}\|v_\varepsilon\|^2)(t)-(\mathbf{y}^\varepsilon,e^\varepsilon)(t)\right\|\leq \varepsilon\, C_{\varepsilon\, t,\|\mathbf{v}^\varepsilon(0)\|,\varepsilon,\|E^\varepsilon\|_{W^{2,\infty}},\|\frac{1}{B^\varepsilon}\|_{W^{1,\infty}}}$$

where $(\mathbf{y}^\varepsilon,e^\varepsilon)$ solves $(\mathbf{y}^\varepsilon,e^\varepsilon)(0)=(\mathbf{x}^\varepsilon,\frac{1}{2}\|v_\varepsilon\|^2)(0)$ and

$$\begin{cases}\dfrac{1}{\varepsilon}\dfrac{\mathrm{d}\mathbf{y}^\varepsilon}{\mathrm{d}t}(t) &= -\dfrac{(\mathbf{E}^\varepsilon)^\perp}{B^\varepsilon}(t,\mathbf{y}^\varepsilon(t))-e^\varepsilon(t)\nabla_{\mathbf{x}}^\perp\left(\dfrac{1}{B^\varepsilon}\right)(t,\mathbf{y}^\varepsilon(t))\\ \dfrac{1}{\varepsilon}\dfrac{\mathrm{d}e^\varepsilon}{\mathrm{d}t}(t) &= e^\varepsilon(t)\,\mathrm{div}_{\mathbf{x}}\left(\dfrac{(\mathbf{E}^\varepsilon)^\perp}{B^\varepsilon}\right)(t,\mathbf{y}^\varepsilon(t))\end{cases}.$$

Inhomogeneous magnetic field, large time.

Proposition

$$\left\|(\mathbf{x}^\varepsilon, \tfrac{1}{2}\|v_\varepsilon\|^2)(t) - (\mathbf{y}^\varepsilon, e^\varepsilon)(t)\right\| \leq \varepsilon\, C_{\varepsilon\, t, \|\mathbf{v}^\varepsilon(0)\|, \varepsilon, \|E^\varepsilon\|_{W^{2,\infty}}, \|\frac{1}{B^\varepsilon}\|_{W^{1,\infty}}}$$

where $(\mathbf{y}^\varepsilon, e^\varepsilon)$ solves $(\mathbf{y}^\varepsilon, e^\varepsilon)(0) = (\mathbf{x}^\varepsilon, \frac{1}{2}\|v_\varepsilon\|^2)(0)$ and

$$\left\{\begin{array}{rcl} \frac{1}{\varepsilon}\frac{\mathrm{d}\mathbf{y}^\varepsilon}{\mathrm{d}t}(t) &=& -\frac{(\mathbf{E}^\varepsilon)^\perp}{B^\varepsilon}(t, \mathbf{y}^\varepsilon(t)) - e^\varepsilon(t)\nabla_{\mathbf{x}}^\perp\left(\frac{1}{B^\varepsilon}\right)(t, \mathbf{y}^\varepsilon(t)) \\ \frac{1}{\varepsilon}\frac{\mathrm{d}e^\varepsilon}{\mathrm{d}t}(t) &=& e^\varepsilon(t)\,\mathrm{div}_{\mathbf{x}}\left(\frac{(\mathbf{E}^\varepsilon)^\perp}{B^\varepsilon}\right)(t, \mathbf{y}^\varepsilon(t)) \end{array}\right. .$$

Remark 1. When B^ε is constant, the equation on $\mathbf{y}^\varepsilon$ uncouples and if moreover $\mathbf{E}$ is curl-free, $\frac{\mathrm{d}e^\varepsilon}{\mathrm{d}t} \equiv 0$.

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Remark 1. When B^ε is constant, the equation on $\mathbf{y}^\varepsilon$ uncouples and if moreover $\mathbf{E}$ is curl-free, $\frac{\mathrm{d}e^\varepsilon}{\mathrm{d}t} \equiv 0$.

Remark 2.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^\varepsilon}{B^\varepsilon(\cdot, \mathbf{y}^\varepsilon)}\right)(t) = -\frac{e^\varepsilon(t)}{B^\varepsilon(t, \mathbf{y}^\varepsilon(t))^2}\left(\partial_t B^\varepsilon + \mathrm{curl}_\mathbf{x}(\mathbf{E}^\varepsilon)\right)(t, \mathbf{y}^\varepsilon(t))$$

thus Faraday's law implies adiabatic invariance of $\mu^\varepsilon := e^\varepsilon / B^\varepsilon(\cdot, \mathbf{y}^\varepsilon)$.

Outline.

General magnetic fields.

Dropping ε-superscripts on fields...

$$\mathbf{B}(t,\mathbf{x}) \,=\, B(t,\mathbf{x})\,\mathbf{e}_{\parallel}(t,\mathbf{x})\,, \qquad B(t,\mathbf{x}) = \|\mathbf{B}(t,\mathbf{x})\|\,.$$

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'Perp' operator

$$\mathbf{J}(t,\mathbf{x})\,\mathbf{a} \,=\, \mathbf{a}\wedge\mathbf{e}_\parallel(t,\mathbf{x})$$

General magnetic fields.

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$$\mathbf{B}(t,\mathbf{x}) = B(t,\mathbf{x})\, \mathbf{e}_\parallel(t,\mathbf{x}), \qquad B(t,\mathbf{x}) = \|\mathbf{B}(t,\mathbf{x})\|.$$

'Perp' operator

$$\mathbf{J}(t,\mathbf{x})\,\mathbf{a} = \mathbf{a} \wedge \mathbf{e}_\parallel(t,\mathbf{x})$$

Decompositions

$$\begin{aligned} v_\parallel(t,\mathbf{x},\mathbf{v}) &= \langle \mathbf{v}\,,\, \mathbf{e}_\parallel(t,\mathbf{x})\rangle\,, \\ \mathbf{v}_\perp(t,\mathbf{x},\mathbf{v}) &= \mathbf{v} - v_\parallel(t,\mathbf{x},\mathbf{v})\; \mathbf{e}_\parallel(t,\mathbf{x})\,, \\ e_\perp(t,\mathbf{x},\mathbf{v}) &= \tfrac{1}{2}\|\mathbf{v}_\perp(t,\mathbf{x},\mathbf{v})\|^2\,, \end{aligned}$$

$$\begin{aligned} E_\parallel(t,\mathbf{x}) &= \langle \mathbf{E}(t,\mathbf{x})\,,\, \mathbf{e}_\parallel(t,\mathbf{x})\rangle\,, \\ \mathbf{E}_\perp(t,\mathbf{x}) &= \mathbf{E}(t,\mathbf{x}) - E_\parallel(t,\mathbf{x})\; \mathbf{e}_\parallel(t,\mathbf{x})\,. \end{aligned}$$

Slow-fast identification.

$$\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t} = \mathbf{v}^\varepsilon\,, \qquad \frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t} = \frac{B(\cdot,\mathbf{x}^\varepsilon)}{\varepsilon}\mathbf{J}(\cdot,\mathbf{x}^\varepsilon)(\mathbf{v}^\varepsilon) + \mathbf{E}(\cdot,\mathbf{x}^\varepsilon)\,.$$

Slow-fast identification.

$$\frac{d\mathbf{x}^\varepsilon}{dt} = \mathbf{v}^\varepsilon, \qquad \frac{d\mathbf{v}^\varepsilon}{dt} = \frac{B(\cdot, \mathbf{x}^\varepsilon)}{\varepsilon}\mathbf{J}(\cdot, \mathbf{x}^\varepsilon)(\mathbf{v}^\varepsilon) + \mathbf{E}(\cdot, \mathbf{x}^\varepsilon)\,.$$

Eliminating $\mathbf{v}_\perp$

$$\mathbf{v}_\perp(\cdot, \mathbf{x}^\varepsilon, \mathbf{v}^\varepsilon) = \varepsilon\frac{\mathbf{J}\mathbf{E}_\perp}{B}(\cdot, \mathbf{x}^\varepsilon) - \varepsilon\frac{\mathbf{J}}{B}(\cdot, \mathbf{x}^\varepsilon)\left(\frac{d\mathbf{v}^\varepsilon}{dt}\right)\,.$$

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Slow variables

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t} &= \mathbf{v}^\varepsilon,\\
\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} &= E_\parallel + \langle \mathbf{v}_\perp, \partial_t\,\mathbf{e}_\parallel + \mathrm{d}_\mathbf{x}\,\mathbf{e}_\parallel\ \mathbf{v}^\varepsilon\rangle\,,\\
\frac{\mathrm{d}e_\perp}{\mathrm{d}t} &= \langle \mathbf{E}_\perp - v_\parallel\ (\partial_t\,\mathbf{e}_\parallel + \mathrm{d}_\mathbf{x}\,\mathbf{e}_\parallel\ \mathbf{v}^\varepsilon)\,, \mathbf{v}_\perp\rangle\,,
\end{aligned}$$

where **evaluations are dropped**.

General magnetic fields, first order.

Theorem (Filbet-LMR, *preprint* 2018.)

$(\mathbf{x}^\varepsilon, v_\parallel(\cdot,\mathbf{x}^\varepsilon,\mathbf{v}^\varepsilon), e_\perp(\cdot,\mathbf{x}^\varepsilon,\mathbf{v}^\varepsilon))$ is $\mathcal{O}(\varepsilon)$ close to $\mathbf{Z}^\varepsilon = (\mathbf{y}^\varepsilon, v^\varepsilon, e^\varepsilon)$ solving

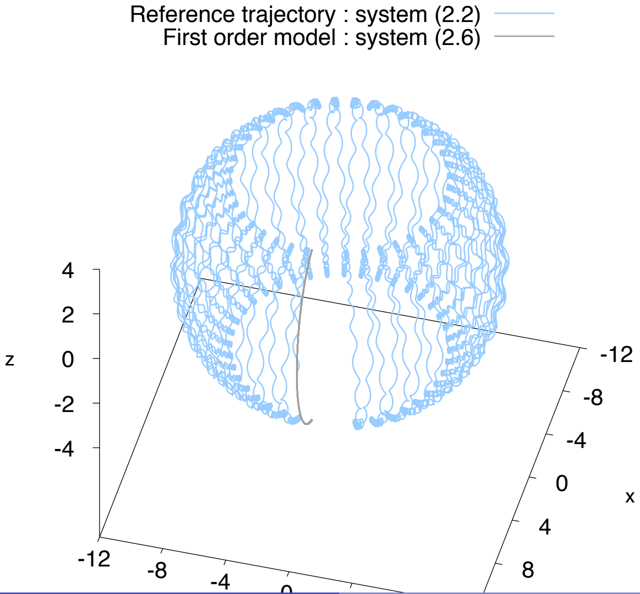
$$\frac{\mathrm{d}\mathbf{Z}^\varepsilon}{\mathrm{d}t}(t) = \mathcal{V}_0(t, \mathbf{Z}^\varepsilon(t))$$

with the same initial datum, where

$$\mathcal{V}_0(t,\mathbf{Z}) = \begin{pmatrix} v\ \mathbf{e}_\parallel(t,\mathbf{y}) \\ E_\parallel(t,\mathbf{y}) + e\ \mathrm{div}_\mathbf{x}\,\mathbf{e}_\parallel(t,\mathbf{y}) \\ -v\,e\ \mathrm{div}_\mathbf{x}\,\mathbf{e}_\parallel(t,\mathbf{y}) \end{pmatrix}.$$

Spatial trajectory, failure of drifts.

Exact *vs.* first-order.



Drifts.

$$\mathbf{U}_{\mathbf{E}\times\mathbf{B}} := \frac{\mathbf{J}\,\mathbf{E}}{B}\,, \qquad \mathbf{U}_{\nabla B\times\mathbf{B}} := \mathbf{J}\nabla_{\mathbf{x}}\left(\frac{1}{B}\right)\,,$$

$$\mathbf{U}_{\partial_t} := -\frac{\mathbf{J}}{B}\,\partial_t\,\mathbf{e}_\parallel\,, \qquad \mathbf{U}_{\mathrm{curl}\,\mathbf{e}_\parallel} := \frac{1}{B}\langle \mathrm{curl}_{\mathbf{x}}\,\mathbf{e}_\parallel\,,\mathbf{e}_\parallel\rangle\;\mathbf{e}_\parallel\,,$$

$$\mathbf{U}_{\mathrm{curv}} := -\frac{\mathbf{J}}{B}(\mathrm{d}_{\mathbf{x}}\,\mathbf{e}_\parallel\;\mathbf{e}_\parallel)\,,$$

$$\begin{aligned} \boldsymbol{\Sigma} &= \mathbf{U}_{\partial_t} + v\,\mathbf{U}_{\mathrm{curv}}\,, \\ \mathbf{U}_{\mathrm{drift}} &= \mathbf{U}_{\mathbf{E}\times\mathbf{B}} + v\,\boldsymbol{\Sigma} + e\,\left(\mathbf{U}_{\mathrm{curl}\,\mathbf{e}_\parallel} + \mathbf{U}_{\nabla B\times\mathbf{B}}\right)\,. \end{aligned}$$

Guiding-center variables.

$$\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t,\mathbf{x},\mathbf{v}) := \mathbf{x} + \varepsilon \frac{\mathbf{J}\,\mathbf{v}}{B}$$

$$v^{\varepsilon}_{\mathrm{gc}}(t,\mathbf{x},\mathbf{v}) := v_{\parallel} + \varepsilon \left\langle \mathbf{v}_{\perp}, \mathbf{\Sigma} \right\rangle + \frac{\varepsilon}{2B} \left\langle \mathbf{J}\,\mathbf{v}_{\perp}, \Re(\mathrm{d}_{\mathbf{x}}\,\mathbf{e}_{\parallel})\,\mathbf{v}_{\perp} \right\rangle$$

$$e^{\varepsilon}_{\mathrm{gc}}(t,\mathbf{x},\mathbf{v}) := e_{\perp} - \varepsilon \left\langle \mathbf{v}_{\perp}, \mathbf{U}_{\mathbf{E}\times\mathbf{B}} + v_{\parallel}\,\mathbf{\Sigma} \right\rangle - \frac{\varepsilon\, v_{\parallel}}{2B} \left\langle \mathbf{J}\,\mathbf{v}_{\perp}, \Re(\mathrm{d}_{\mathbf{x}}\,\mathbf{e}_{\parallel})\,\mathbf{v}_{\perp} \right\rangle$$

General magnetic fields, second order.

Theorem (Filbet-LMR, *preprint* 2018.)

$(\mathbf{x}^{\varepsilon}_{\rm gc}, v_{\rm gc}(\cdot, \mathbf{x}^{\varepsilon}, \mathbf{v}^{\varepsilon}), e_{\rm gc}(\cdot, \mathbf{x}^{\varepsilon}, \mathbf{v}^{\varepsilon}))$ is $\mathcal{O}(\varepsilon^2)$ close to $\mathbf{Z}^{\varepsilon}_{\rm gc}$ solving

$$\frac{\mathrm{d}\mathbf{Z}^{\varepsilon}_{\rm gc}}{\mathrm{d}t}(t) = (\mathcal{V}_0 + \varepsilon\,\mathcal{V}_1)(t, \mathbf{Z}^{\varepsilon}_{\rm gc}(t))$$

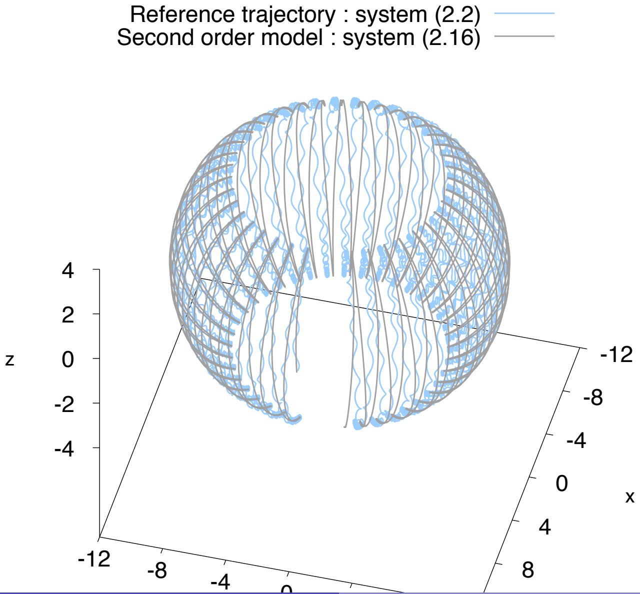
with the same initial datum, where

$$\mathcal{V}_1 = \begin{pmatrix} \mathbf{U}_{\rm drift} \\ \langle \mathbf{\Sigma}, \mathbf{E}\rangle + e\ \mathrm{div}_{\mathbf{x}}\,\mathbf{\Sigma} \\ -e\ [\,\langle \mathbf{U}_{\rm curv}, \mathbf{E}\rangle + \mathrm{div}_{\mathbf{x}}\,(\mathbf{U}_{\mathbf{E}\times\mathbf{B}} + v\mathbf{\Sigma})\,] \end{pmatrix}.$$

For an example of large time, see also Filbet-LMR, *preprint* 2018.

Illustration: spatial trajectory.

Exact *vs.* second-order.

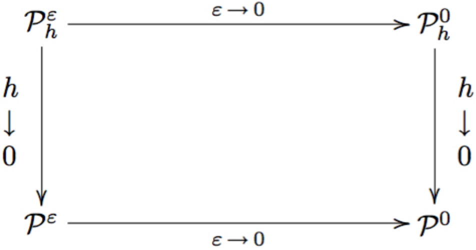


Outline.

Outline.

Preserving asymptotics.

h discretization parameter.
ε asymptotic parameter.



Goals.

Main goals

- $\mathcal{P}_h^\varepsilon \to \mathcal{P}_h^0$ uniformly in h:
  compute slow dynamics even with a low resolution.

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Extra constraints

- Non invasive and robust.

- Higher-order.

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Extra constraints

- Non invasive and robust.

- Higher-order.

Similar goals in

Frénod-Hirstoaga-Lutz-Sonnendrücker, *Commun. Comput. Phys.* 2015.

Computation of a few oscillation phases in

Crouseilles-Lemou-Méhats, *JCP* 2013.

Particle-in-cell methods.

Fix a time step Δt. For some $m \in \mathbf{N}^*$, f_0 approximated by

$$f_m^0 = \sum_{k=0}^{m} \omega_k^m \, \delta_{\mathbf{X}_m^k} \, .$$

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$$f_m^0 = \sum_{k=0}^{m} \omega_k^m \, \delta_{\mathbf{X}_m^k} .$$

Then, for $n \in \mathbf{N}$,

$$f_\varepsilon^{n,\Delta t,m} := (\Phi_\varepsilon^{n,\Delta t,m})_*(f_m^0) = \sum_{k=0}^{m} \omega_k^m \, \delta_{\Phi_\varepsilon^{n,\Delta t,m}(\mathbf{X}_m^k)} .$$

where $\Phi_\varepsilon^{n,\Delta t,m}$ is a discrete flow approximating $\Phi^\varepsilon(n\Delta t, 0, \cdot)$.

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where $\Phi_\varepsilon^{n,\Delta t,m}$ is a discrete flow approximating $\Phi^\varepsilon(n\Delta t, 0, \cdot)$.

Remark. For nonlinear PDEs, everything is computed altogether and Dirac masses are smoothed.

For Vlasov-Poisson, see Filbet-LMR, *SIAM Num. Anal.* 2016

and Filbet-LMR, *SIAM Num. Anal.* 2017.

Outline.

Long time variables and slow fields.

From now on, we restrict to dimension two and large time asymptotics.

Long time variables and slow fields.

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For the sake of clarity, we replace t with $\varepsilon\, t$

$$
\begin{aligned}
\varepsilon\,\frac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t}(t) &= \mathbf{v}^\varepsilon(t)\\
\varepsilon\,\frac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t}(t) &= -\frac{B(t,\mathbf{x}^\varepsilon(t))}{\varepsilon}\,(\mathbf{v}^\varepsilon(t))^\perp + \mathbf{E}(t,\mathbf{x}^\varepsilon(t))
\end{aligned}
$$

hence **implicitly** assuming slow fields.

Homogeneous case: a first-order scheme.

$(\Phi_\varepsilon^{n,\Delta t}(\mathbf{x}_\varepsilon^0, \mathbf{v}_\varepsilon^0))_{n\in\mathbf{N}}$ computed through

$$\begin{aligned}\frac{\mathbf{x}_\varepsilon^{n+1,\Delta t} - \mathbf{x}_\varepsilon^{n,\Delta t}}{\Delta t} &= \frac{\mathbf{v}_\varepsilon^{n+1,\Delta t}}{\varepsilon},\\ \frac{\mathbf{v}_\varepsilon^{n+1,\Delta t} - \mathbf{v}_\varepsilon^{n,\Delta t}}{\Delta t} &= \frac{1}{\varepsilon}\left(-\frac{(\mathbf{v}_\varepsilon^{n+1,\Delta t})^\perp}{\varepsilon} + \mathbf{E}(t_{\Delta t}^n, \mathbf{x}_\varepsilon^{n,\Delta t})\right).\end{aligned}$$

where $t_{\Delta t}^n = n\Delta t$.

Inspired by Boscarino-Filbet-Russo, *J. Sci. Comput.* 2016
on dissipative problems.

Homogeneous case: a first-order scheme.

$(\Phi_\varepsilon^{n,\Delta t}(\mathbf{x}_\varepsilon^0, \mathbf{v}_\varepsilon^0))_{n\in\mathbf{N}}$ computed through

$$\begin{aligned}\frac{\mathbf{x}^{n+1} - \mathbf{x}^n}{\Delta t} &= \frac{\mathbf{v}^{n+1}}{\varepsilon},\\ \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} &= \frac{1}{\varepsilon}\left(-\frac{(\mathbf{v}^{n+1})^\perp}{\varepsilon} + \mathbf{E}(t^n, \mathbf{x}^n)\right).\end{aligned}$$

where $t^n = n\Delta t$.

Implicit dependences on ε and Δt.

Inspired by Boscarino-Filbet-Russo, *J. Sci. Comput.* 2016 on dissipative problems.

Consistency when $\varepsilon \to 0$ at fixed Δt.

Proposition (Filbet-LMR, *SIAM Num. Anal.* 2016.)

Fix $\Delta t > 0$. Assume

- for any $n \in \mathbf{N}$, $\varepsilon^2 \mathbf{x}_\varepsilon^n \stackrel{\varepsilon \to 0}{\to} 0$;
- $(\mathbf{x}_\varepsilon^0, \varepsilon\, \mathbf{v}_\varepsilon^0) \stackrel{\varepsilon \to 0}{\to} (\mathbf{y}^0, 0)$.

Then, for any n, there exists $\mathbf{y}^n$ such that

$$\mathbf{x}_\varepsilon^n \stackrel{\varepsilon \to 0}{\to} \mathbf{y}^n\,.$$

Limiting sequence satisfies

$$\frac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = -\mathbf{E}(t^n, \mathbf{y}^n)^\perp\,.$$

From now on, fixed initial data.

Convergence analysis, spatial variable.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\|\mathbf{x}^{\varepsilon}(t^n) - \mathbf{y}(t_n)\| = \mathcal{O}(\varepsilon), \qquad ,$$

$$\|\mathbf{y}(t^n) - \mathbf{y}^n\| = \mathcal{O}(\Delta t),$$

Convergence analysis, spatial variable.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\|\mathbf{x}^\varepsilon(t^n) - \mathbf{y}(t_n)\| = \mathcal{O}(\varepsilon), \qquad \|\mathbf{x}_\varepsilon^n - \mathbf{y}^n\| = \mathcal{O}(\varepsilon),$$

$$\|\mathbf{y}(t^n) - \mathbf{y}^n\| = \mathcal{O}(\Delta t), \ \|\mathbf{x}_\varepsilon^n - \mathbf{x}^\varepsilon(t_n)\| = \mathcal{O}\left(\min\left(\left\{\Delta t + \varepsilon, \frac{\Delta t}{\varepsilon^3}\right\}\right)\right).$$

Convergence analysis, spatial variable.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\|\mathbf{x}^\varepsilon(t^n)-\mathbf{y}(t_n)\| = \mathcal{O}(\varepsilon), \qquad \|\mathbf{x}^n_\varepsilon-\mathbf{y}^n\| = \mathcal{O}(\varepsilon),$$

$$\|\mathbf{y}(t^n)-\mathbf{y}^n\| = \mathcal{O}(\Delta t), \ \|\mathbf{x}^n_\varepsilon-\mathbf{x}^\varepsilon(t_n)\| = \mathcal{O}\left(\min\left(\left\{\Delta t+\varepsilon, \frac{\Delta t}{\varepsilon^3}\right\}\right)\right).$$

Thus

$$\|\mathbf{x}^n_\varepsilon-\mathbf{x}^\varepsilon(t_n)\| = \begin{cases} \mathcal{O}(\Delta t) & \text{when } \varepsilon \leq \Delta t \\ \mathcal{O}(\varepsilon) & \text{when } \varepsilon^4 \leq \Delta t \leq \varepsilon \\ \mathcal{O}\left(\frac{\Delta t}{\varepsilon^3}\right) & \text{when } \Delta t \leq \varepsilon^4 \end{cases}$$

and in particular $\|\mathbf{x}^n_\varepsilon-\mathbf{x}^\varepsilon(t_n)\| = \mathcal{O}((\Delta t)^{\frac{1}{4}})$.

Convergence analysis, guiding center.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\|\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t^n) - \mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t_n)\| = \mathcal{O}(\varepsilon^2),$$

,

$$\|\mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t^n) - (\mathbf{y}_{\mathrm{gc}})^n_{\varepsilon}\| = \mathcal{O}(\Delta t),$$

Convergence analysis, guiding center.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\begin{aligned}
\|\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t^n) - \mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t_n)\| &= \mathcal{O}(\varepsilon^2),\\
\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon} - (\mathbf{y}_{\mathrm{gc}})^n_{\varepsilon}\| &= \mathcal{O}(\varepsilon^2 + \Delta t\,\varepsilon),\\
\|\mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t^n) - (\mathbf{y}_{\mathrm{gc}})^n_{\varepsilon}\| &= \mathcal{O}(\Delta t),\\
\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon} - \mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t_n)\| &= \mathcal{O}\left(\min\left(\left\{\Delta t + \varepsilon^2, \frac{\Delta t}{\varepsilon^3}\right\}\right)\right).
\end{aligned}$$

Convergence analysis, guiding center.

Proposition (Filbet-LMR-Zakerzadeh, *work in progress*.)

$$\|\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t^n)-\mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t_n)\| = \mathcal{O}(\varepsilon^2),$$

$$\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon}-(\mathbf{y}_{\mathrm{gc}})^n_{\varepsilon}\| = \mathcal{O}(\varepsilon^2+\Delta t\,\varepsilon),$$

$$\|\mathbf{y}^{\varepsilon}_{\mathrm{gc}}(t^n)-(\mathbf{y}_{\mathrm{gc}})^n_{\varepsilon}\| = \mathcal{O}(\Delta t),$$

$$\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon}-\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t_n)\| = \mathcal{O}\left(\min\left(\left\{\Delta t+\varepsilon^2, \frac{\Delta t}{\varepsilon^3}\right\}\right)\right).$$

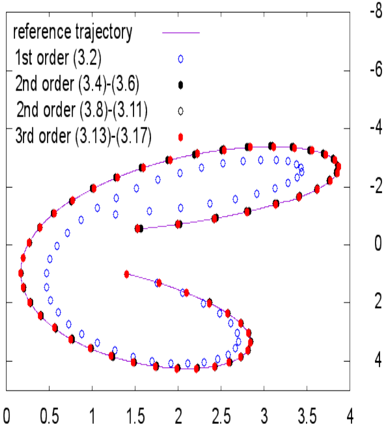
Thus

$$\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon}-\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t_n)\| = \begin{cases}\mathcal{O}(\Delta t) & \text{when } \varepsilon^2\leq\Delta t\\ \mathcal{O}(\varepsilon^2) & \text{when } \varepsilon^5\leq\Delta t\leq\varepsilon^2\\ \mathcal{O}\left(\frac{\Delta t}{\varepsilon^3}\right) & \text{when } \Delta t\leq\varepsilon^5\end{cases}$$

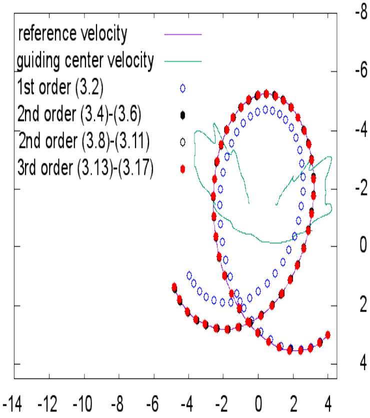
and in particular $\|(\mathbf{x}_{\mathrm{gc}})^n_{\varepsilon}-\mathbf{x}^{\varepsilon}_{\mathrm{gc}}(t_n)\| = \mathcal{O}((\Delta t)^{\frac{2}{5}})$.

Phase space, $\varepsilon = 0.1$.

(a) spatial trajectory,

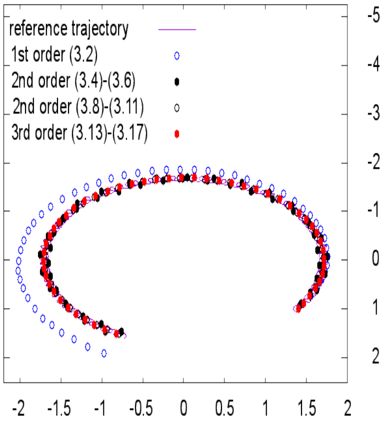


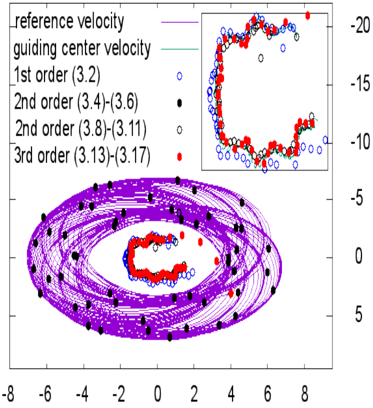
(b) velocity trajectory.



Phase space, $\varepsilon = 0.01$.

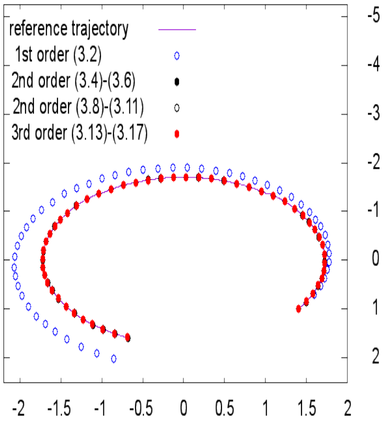
(a) spatial trajectory, (b) velocity trajectory.



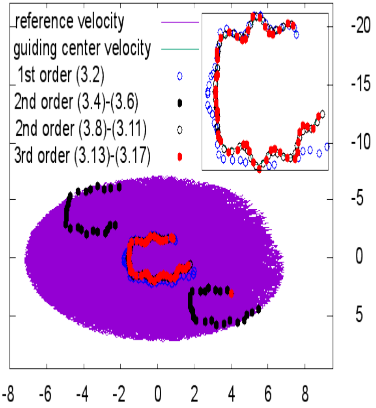


Phase space, $\varepsilon = 0.001$.

(a) spatial trajectory,



(b) velocity trajectory.



Outline.

1. Elements of gyrokinetics
2. A few asymptotic preserving shemes
   - Framework
   - Homogenous magnetic fields
   - Inhomogeneous fields
3. Conclusion

Continuous systems.

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t} &= \mathbf{v}^\varepsilon \\ \varepsilon \dfrac{\mathrm{d}\mathbf{v}^\varepsilon}{\mathrm{d}t} &= -\dfrac{B(t,\mathbf{x}^\varepsilon(t))}{\varepsilon}(\mathbf{v}^\varepsilon)^\perp + \mathbf{E}(t,\mathbf{x}^\varepsilon(t)) \end{cases}.$$

$(\mathbf{x}^\varepsilon, \frac{1}{2}\|\mathbf{v}^\varepsilon\|^2)$ converges to $(\mathbf{y}, e)$ such that

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}(t) &= -\dfrac{\mathbf{E}^\perp}{B}(t,\mathbf{y}(t)) - e(t)\nabla_{\mathbf{x}}^\perp\left(\dfrac{1}{B}\right)(t,\mathbf{y}(t)) \\ \dfrac{\mathrm{d}e}{\mathrm{d}t}(t) &= e(t)\,\mathrm{div}_{\mathbf{x}}\left(\dfrac{\mathbf{E}^\perp}{B}\right)(t,\mathbf{y}(t)) \end{cases}.$$

Issue

How to get $\mathbf{v}^\varepsilon$ go to 0 weakly and $\|\mathbf{v}^\varepsilon\|^2$ go to $2e$ strongly at discrete level ?

Augmented version.

An answer. Discretize

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t} & = & \mathbf{w}^\varepsilon \\ \varepsilon \dfrac{\mathrm{d}e^\varepsilon}{\mathrm{d}t} & = & \langle \mathbf{E}, \mathbf{w}^\varepsilon \rangle \\ \varepsilon \dfrac{\mathrm{d}\mathbf{w}^\varepsilon}{\mathrm{d}t} & = & \dfrac{B}{\varepsilon} (\mathbf{w}^\varepsilon)^\perp + \mathbf{E} \end{cases}$$

and recover $\mathbf{v}^\varepsilon$ by

$$\mathbf{v}^\varepsilon = \sqrt{2}e^\varepsilon \frac{\mathbf{w}^\varepsilon}{\|\mathbf{w}^\varepsilon\|}.$$

Augmented version.

An answer. Discretize

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}\mathbf{x}^\varepsilon}{\mathrm{d}t} &= \mathbf{w}^\varepsilon \\ \varepsilon \dfrac{\mathrm{d}e^\varepsilon}{\mathrm{d}t} &= \langle \mathbf{E}, \mathbf{w}^\varepsilon \rangle \\ \varepsilon \dfrac{\mathrm{d}\mathbf{w}^\varepsilon}{\mathrm{d}t} &= \dfrac{B}{\varepsilon} (\mathbf{w}^\varepsilon)^\perp + \mathbf{E} - \chi(e^\varepsilon, \mathbf{w}^\varepsilon)\, \nabla_x(\ln(B)) \end{cases}$$

and recover $\mathbf{v}^\varepsilon$ by

$$\mathbf{v}^\varepsilon = \sqrt{2}e^\varepsilon \frac{\mathbf{w}^\varepsilon}{\|\mathbf{w}^\varepsilon\|}\,.$$

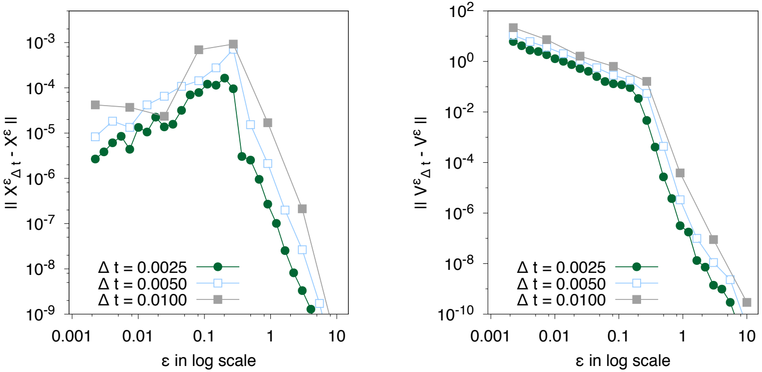
χ chosen so that

$$\chi\left(\tfrac{1}{2}\|\mathbf{w}\|^2, \mathbf{w}\right) = 0\,, \qquad \lim_{\mathbf{w}\to 0} \chi(e, \mathbf{w}) = e\,.$$

See Filbet-LMR, *SIAM Num. Anal.* 2017.

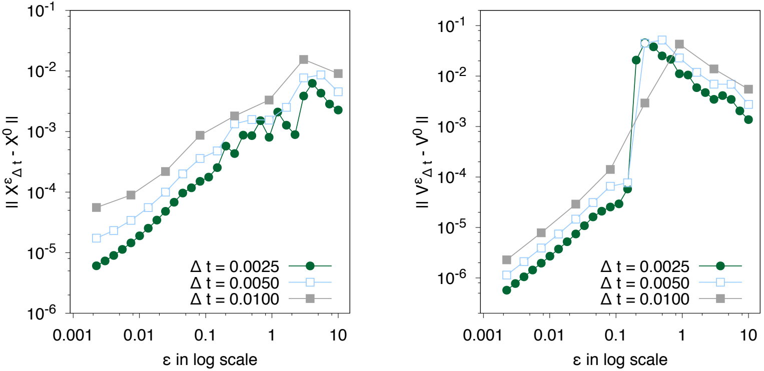
Accuracy.

Errors (a) $\|\mathbf{x}_{\varepsilon}^{\Delta t} - \mathbf{x}^{\varepsilon}\|$, (b) $\|\mathbf{v}_{\varepsilon}^{\Delta t} - \mathbf{v}^{\varepsilon}\|$ for a third order scheme.

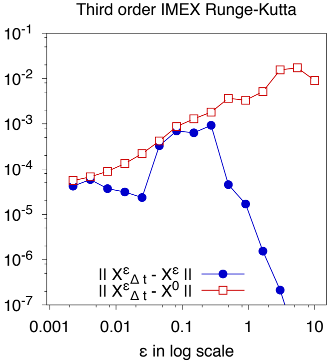


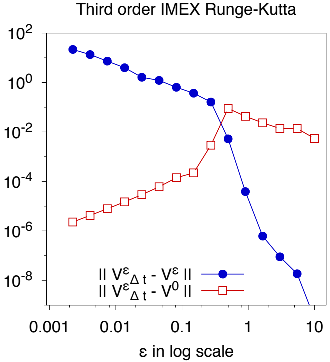
Asymptotics.

Errors (a) $\|\mathbf{x}_{\varepsilon}^{\Delta t} - \mathbf{y}\|$, (b) $\|\mathbf{v}_{\varepsilon}^{\Delta t} - "\mathbf{v}"\|$ for a third order scheme.



Transitions.





Superimposed comparisons.

Outline.

References. Filbet-LMR,

- *Asymptotically stable PIC methods for the VP system with strong external magnetic field. SIAM Num. Anal.* 2016.
- *Asymptotically preserving particle-in-cell methods for inhomogeneous strongly magnetized plasmas. SIAM Num. Anal.* 2017.
- *Asymptotics of the three dimensional Vlasov equation in the large magnetic field limit. preprint* 2018.
- *Convergence analysis of asymptotically preserving schemes for strongly magnetized plasmas. Work in progress*. With Zakerzadeh

Analytic prospects.

- **Large time** complex **3-D** geometries.
- **Nonlinear fields**.
- Interaction with **evanescent collisions**. See work with Herda.

Numerical prospects.

- **Complex 3-D** geometries.