

# Asymptotic and null controllability of an advection-diffusion equation

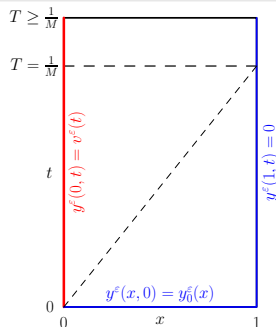
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Laboratoire de mathématiques Blaise Pascal

Workshop *Qualitative behavior of kinetic equations and related problems*  
Hausdorff research institute for Mathematics - Bonn - June 2019



Let  $T > 0$ ,  $M \in \mathbb{R}^*$ ,  $\varepsilon > 0$  and  $Q_T := (0, 1) \times (0, T)$ .

$$\begin{cases} L_\varepsilon y^\varepsilon := y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & \text{in } Q_T, \\ y^\varepsilon(0, \cdot) = v^\varepsilon(t), y^\varepsilon(1, \cdot) = 0, & \text{in } (0, T), \\ y^\varepsilon(\cdot, 0) = y_0^\varepsilon, & \text{in } (0, 1). \end{cases} \quad (1)$$



- **Well-posedness:**

$$\forall y_0^\varepsilon \in H^{-1}(0, 1), v^\varepsilon \in L^2(0, T), \quad \exists! y^\varepsilon \in L^2(Q_T) \cap C([0, T]; H^{-1}(0, 1))$$

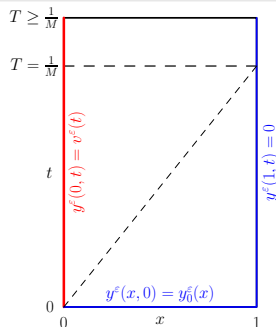
- **Null controllability property:** From [Fursikov'91],

$$\forall T > 0, y_0^\varepsilon \in H^{-1}(0, 1), \exists v^\varepsilon \in L^2(0, T) \quad \text{s.t.} \quad y^\varepsilon(\cdot, T) = 0 \quad \text{in } H^{-1}(0, 1)$$

- **Main concern:** Behavior of the controls  $v^\varepsilon$  as  $\varepsilon \rightarrow 0$
- **Remark:**  $y^\varepsilon$  exhibits internal and boundary layers as  $\varepsilon \rightarrow 0$  and make non trivial the analysis of the direct and control problems !

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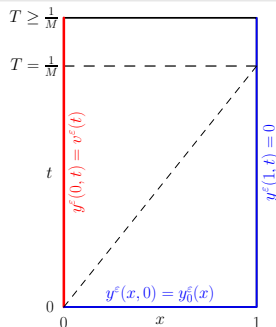
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This 1d scalar linear eq. is an **embedded system of the Navier-Stokes system** with non-characteristic boundary condition and viscosity coefficient equals to  $\varepsilon$  as discussed in [[Cheng-Temam-Wang' 2000](#)]<sup>1</sup>

and appears in many contexts, notably

- **Asymptotic analysis of the direct problem:**  
[L. Bobisud](#): *Second-order linear parabolic equations with a small parameter*, Arch. Rational Mech. Anal., (1967).
- **Numerical approximation of the solution:**  
[P. Deuring](#), [R. Eymard](#), and [M. Mildner](#) :  *$L^2$ -stability independent of diffusion for a finite element-finite volume discretization of a linear convection-diffusion equation*, SIAM J. Numer. Anal (2015).
- **Controllability:**  
[J.-M. Coron](#), [S. Guerrero](#): *Singular optimal control: a linear 1-D parabolic-hyperbolic example*, Asymptot. Anal., (2005).

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<sup>1</sup> New approximation algorithms for a class of partial differential equations displaying boundary layer behavior, Methods Appl. Anal. (2000)

- We note the non empty set of null controls by

$$\mathcal{C}(y_0^\varepsilon, T, \varepsilon, M) := \left\{ v \in L^2(0, T); y^\varepsilon = y^\varepsilon(v^\varepsilon) \text{ solves (9) and satisfies } y^\varepsilon(\cdot, T) = 0 \right\}$$

and define the **cost of control** by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0, T)} \right\}.$$

$K(\varepsilon, T, M)$  is the norm of the (linear) operator  $y_0^\varepsilon \rightarrow v_{HUM}$  where  $v_{HUM}$  is the control of minimal  $L^2$ -norm

- We denote

$$T_M := \inf \left\{ T > 0; \sup_{\varepsilon > 0} K(\varepsilon, T, M) < \infty \right\}$$

- **Remark**  $K(\varepsilon, T, 0) \sim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1/2} e^{\frac{\kappa}{\varepsilon} T}$ ,  $\kappa \in (1/2, 3/4)$  so that  $T_0 = \infty$  [Miller'06].  
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With respect to the null controllability issue

- There is a kind of **competition between the transport and the diffusion terms**: as  $\varepsilon \rightarrow 0$ , the transport term becomes dominant, pushes the solution out of  $(0, 1)$  and makes  $\|y^\varepsilon(\cdot, T)\|_2$  small for all  $T \geq 1/|M|$ . However, as  $\varepsilon \rightarrow 0$ , the diffusion term, which is the main tools to control to zero the solution, is small.

Intuitively, one have to wait enough time, from  $t = 1/|M|$ , to control uniformly w.r.t.  $\varepsilon$  the remainder  $y^\varepsilon(\cdot, 1/|M|)$ .

- The negative case  $M < 0$  is the "most singular" since then the transport term pushes the solution  $y^\varepsilon$  from the right to the left line  $x = 0$  where the control acts. **The control requires more "energy" to act on the whole spatial domain.**

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**Main objective** : Determine the behavior of the cost  $K(\varepsilon, T, M)$  as  $\varepsilon \rightarrow 0$

**Outline** :

- Part 1: Facts on the diffusion-advection eq. and literature.
- Part 2: Numerical attempt to estimate  $K(\varepsilon, T, M)$ .
- Part 3: Asymptotic analysis of the corresponding optimality system

## Remark

- By duality, the controllability property of (9) is related to the existence of a constant  $C > 0$  such that

$$\|\varphi(\cdot, 0)\|_{L^2(0,1)} \leq C \|\varepsilon \varphi_x(0, \cdot)\|_{L^2(0,T)}, \quad \forall \varphi_T \in H_0^1(0,1) \cap H^2(0,1) \quad (2)$$

where  $\varphi$  solves the adjoint system

$$\begin{cases} L_\varepsilon^* \varphi := \varphi_t + \varepsilon \varphi_{xx} + M \varphi_x = 0 & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(1, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_T & \text{in } (0, 1). \end{cases}$$

- The quantity

$$C_{obs}(\varepsilon, T, M) = \sup_{\varphi_T \in H_0^1(0,1)} \frac{\|\varphi(\cdot, 0)\|_{L^2(0,1)}}{\|\varepsilon \varphi_x(0, \cdot)\|_{L^2(0,T)}}$$

is the smallest constant for which (2) holds true and

$$K(\varepsilon, T, M) = C_{obs}(\varepsilon, T, M).$$

## Theorem (Coron-Guerrero, 2005)

Let  $T > 0$ ,  $M \in \mathbb{R}^*$ ,  $y_0 \in L^2(0, 1)$  **independent of  $\varepsilon$** . Let  $(v^\varepsilon)_{(\varepsilon)}$  be a sequence of functions in  $L^2(0, T)$  such that for some  $v \in L^2(0, T)$

$$v^\varepsilon \rightharpoonup v \text{ in } L^2(0, T), \text{ as } \varepsilon \rightarrow 0^+.$$

For  $\varepsilon > 0$ , let us denote by  $y^\varepsilon \in C([0, T]; H^{-1}(0, 1))$  the weak solution of

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0 & Q_T, \\ y^\varepsilon(0, \cdot) = v^\varepsilon, y^\varepsilon(1, \cdot) = 0 & (0, T), \\ y^\varepsilon(\cdot, 0) = y_0 & (0, 1). \end{cases}$$

Let  $y \in C([0, T]; L^2(0, 1))$  be the weak solution of

$$\begin{cases} y_t + M y_x = 0 & Q_T, \\ y(0, \cdot) = v \text{ if } M > 0 & (0, T), \\ y(1, \cdot) = 0 \text{ if } M < 0 & (0, T), \\ y(\cdot, 0) = y_0 & (0, 1). \end{cases}$$

Then,  $y^\varepsilon \rightharpoonup y$  in  $L^2(Q_T)$  as  $\varepsilon \rightarrow 0^+$ .

## Corollary

For  $T < \frac{1}{|M|}$ ,  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon, T, M) \rightarrow \infty$ . Consequently,  $T_M \geq \frac{1}{|M|}$ .

**PROOF.** Assume that  $K(\varepsilon, T, M) \not\rightarrow +\infty$ . There exists  $(\varepsilon_n)_{(n \in \mathbb{N})}$  positive tending to 0 such that  $(K(\varepsilon_n, T, M))_{(n \in \mathbb{N})}$  is bounded.

Let  $v^{\varepsilon_n}$  the optimal control driving  $y_0$  to 0 at time  $T$  and  $y^{\varepsilon_n}$  the corresponding solution. Let  $T_0 \in (T, 1/|M|)$ . We extend  $y^{\varepsilon_n}$  and  $v^{\varepsilon_n}$  by 0 on  $(T, T_0)$ . From the inequality

$$\|v^{\varepsilon_n}\|_{L^2(0, T_0)} = \|v^{\varepsilon_n}\|_{L^2(0, T)} \leq K(\varepsilon_n, T, M) \|y_0\|_{L^2(0, 1)},$$

we deduce that  $(v^{\varepsilon_n})_{(n \in \mathbb{N})}$  is bounded in  $L^2(0, T_0)$ , so we extract a subsequence  $(v^{\varepsilon_{n_k}})_{(k \in \mathbb{N})}$  such that  $v^{\varepsilon_{n_k}} \rightharpoonup v$  in  $L^2(0, T_0)$ . We deduce that  $y^{\varepsilon_{n_k}} \rightharpoonup y$  in  $L^2(Q_{T_0})$  solution of the transport equation. Necessarily,  $y \equiv 0$  on  $(0, 1) \times (T, T_0)$ . **Contradiction.**

# Exponential blow up of the control cost for $M > 0$

Theorem (Coron-Guerrero'2005)

• Assume  $M > 0, T < 1/M$ .  $\exists c, C > 0$  such that  $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon} \forall \varepsilon > 0$ .

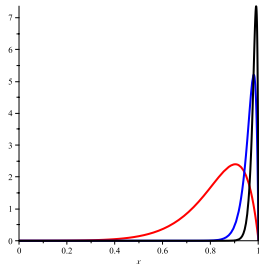
The lower bound are obtained using **specific initial condition**:

$$y_0^\varepsilon(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x),$$

$$K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2} e^{-\frac{M}{2\varepsilon}}) \quad \text{s.t.} \quad \|y_0^\varepsilon\|_{L^2(0,1)} = 1$$

leading, for  $M > 0$ , to

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon} (1 - TM) - \pi^2 \varepsilon T\right)$$



$y_0^\varepsilon$  for  $\varepsilon = 5 \times 10^{-2}$ ,  
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$\Rightarrow$  Surprising result since  $\forall \varepsilon > 0, t > 0, \|y^\varepsilon(\cdot, t)\|_2 = \mathcal{O}(e^{-t/\varepsilon})$  !!!

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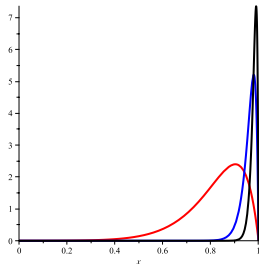
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## Theorem (Coron-Guerrero'2005)

- If  $M < 0$ , then  $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$ ,  $c, C > 0$ , when  $\varepsilon \rightarrow 0$  for  $T < \frac{2}{|M|}$ .

With again  $y_0(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x)$ ,

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The negative case  $M < 0$  is more singular, since the transport term acts "against" the control function.

## Lemma

Let  $\alpha \in [0, 1)$ . The **free solution** (i.e.  $v^\varepsilon \equiv 0$ ) satisfies

$$\|y^\varepsilon(\cdot, t)\|_{L^2(0,1)} \leq \|y_0^\varepsilon\|_{L^2(0,1)} e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}t}, \quad \forall t \geq \frac{1}{|M|(1-\alpha)}.$$

**PROOF.** Assume  $M > 0$ .  $z^\varepsilon(x, t) = e^{-\frac{M\alpha x}{2\varepsilon}t} y^\varepsilon(x, t)$  solves

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and the result since  $t \geq \frac{1}{M(1-\alpha)}$  implies  $\frac{M\alpha}{2}(1 - Mt + \frac{M\alpha t}{2}) \leq -\frac{M\alpha^2}{4(1-\alpha)}$ . □

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## Corollary (Cost of approximate control)

$$\forall \delta > 0, \forall T > \frac{1}{|M|}, \quad \exists \varepsilon_0(\delta) > 0 \quad \text{s.t.} \quad \forall \varepsilon < \varepsilon_0, \quad K_\delta(\varepsilon, T, M) = 0 \quad (3)$$

where

$$\mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M) := \left\{ v \in L^2(0, T); y = y(v) \text{ solves (9) and satisfies } \|y(\cdot, T)\|_{L^2(0,1)} \leq \delta \right\}$$

and

$$K_\delta(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0, T)} \right\}.$$

## Lemma

Let  $\alpha \in [0, 1)$ . The *free solution* (i.e.  $v^\varepsilon = 0$ ) satisfies

$$\|y^\varepsilon(\cdot, t)\|_{L^2(0,1)} \leq \|y_0^\varepsilon\|_{L^2(0,1)} e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}t}, \quad \forall t \geq \frac{1}{|M|(1-\alpha)}.$$

## Corollary (Cost of approximate control)

$$\forall \delta > 0, \forall T > \frac{1}{|M|}, \quad \exists \varepsilon_0(\delta) > 0 \quad \text{s.t.} \quad \forall \varepsilon < \varepsilon_0, \quad K_\delta(\varepsilon, T, M) = 0 \quad (3)$$

where

$$\mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M) := \left\{ v \in L^2(0, T); y = y(v) \text{ solves (9) and satisfies } \|y(\cdot, T)\|_{L^2(0,1)} \leq \delta \right\}$$

and

$$K_\delta(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}_\delta(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0, T)} \right\}.$$

### Proposition (Amirat, M 19)

Assume  $M > 0$ ,  $v^\varepsilon \equiv 0$ ,  $y_0^\varepsilon = y_0 \in H^3(0, 1)$ . For  $\varepsilon > 0$  small enough, the free solution  $y^\varepsilon$  satisfies

$$\left\| y^\varepsilon \left( \cdot, \frac{1}{M} \right) \right\|_{L^2(0,1)} \leq c \left( |y_0(0)|\varepsilon^{1/4} + |y_0^{(1)}(0)|\varepsilon^{3/4} + |y_0^{(2)}(0)|\varepsilon^{5/4} \right) + \mathcal{O}(\varepsilon^{3/2}) \quad (4)$$

for some constant  $c > 0$ , independent of  $\varepsilon$ .

## Theorem (Coron-Guerrero'2005)

- Assume  $M > 0$  and  $T \geq \frac{4.3}{M}$ .  $\exists c, C > 0$  s.t.  $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon} \forall \varepsilon > 0$ .
- Assume  $M < 0$  and  $T \geq \frac{57.2}{M}$ .  $\exists c, C > 0$  s.t.  $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon} \forall \varepsilon > 0$ .



## Theorem (Coron-Guerrero'2005)

$$T_M \in [1, 4.3] \frac{1}{M} \text{ if } M > 0, \quad [2, 57.2] \frac{1}{|M|} \text{ if } M < 0.$$

## Theorem (Glass'2009)

$$T_M \in [1, 4.2] \frac{1}{M} \text{ if } M > 0, \quad [2, 6.1] \frac{1}{|M|} \text{ if } M < 0.$$

## Theorem (Lissy'2015)

$$T_M \in [1, 2\sqrt{3}] \frac{1}{M} \text{ if } M > 0, \quad [2\sqrt{2}, 2(1 + \sqrt{3})] \frac{1}{|M|} \text{ if } M < 0.$$

$$(2\sqrt{3} \approx 3.46)$$

## Theorem (Darde-Ervedoza'2017)

$$T_M \in [1, K] \frac{1}{M} \text{ if } M > 0, \quad K \approx 3.34$$

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Numerical estimate of the cost  $K(\varepsilon, T, M)$  w.r.t.  $\varepsilon$  !??

**Rk:** At the numerical level, with  $\delta$  small, one can not distinguish the  $\delta$  approximate controllability from the null controllability. Therefore, the numerical estimate is a priori feasible only for "large"  $\varepsilon$  and  $T$  "close" to  $1/|M|$ .

## Reformulation of the cost of control

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

where  $\mathcal{A}_\varepsilon : L^2(0, 1) \rightarrow L^2(0, 1)$  is the **control operator** defined by  $\mathcal{A}_\varepsilon y_0 := -\hat{\varphi}(0)$  where  $\hat{\varphi}$  solves the adjoint system

$$\begin{cases} L_\varepsilon^* \varphi := \varphi_t + \varepsilon \varphi_{xx} + M \varphi_x = 0 & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(1, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_T & \text{in } (0, 1), \end{cases}$$

associated to the initial condition  $\varphi_T \in H_0^1(0, 1)$ , solution of the extremal problem

$$\inf_{\varphi_T \in H_0^1(0,1)} J^*(\varphi_T) := \frac{1}{2} \|\varepsilon \varphi_x(0, \cdot)\|_{L^2(0,T)}^2 + (y_0, \varphi(\cdot, 0))_{L^2(0,1)}.$$

**REFORMULATION** -  $K(\varepsilon, T, M)$  is solution of the **generalized eigenvalue problem** :

$$\sup \left\{ \sqrt{\lambda} \in \mathbb{R} : \exists y_0 \in L^2(0, 1), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \text{ in } L^2(0, 1) \right\}.$$

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

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# The generalized eigenvalue problem by the power iterated method

In order to get the largest eigenvalue of the operator  $\mathcal{A}_\varepsilon$ , we may employ the **power iterate method** (Chatelain'89):

$$\begin{cases} y_0^0 \in L^2(0,1) \text{ given such that } \|y_0^0\|_{L^2(0,1)} = 1, \\ \tilde{y}_0^{k+1} = \mathcal{A}_\varepsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \frac{\tilde{y}_0^{k+1}}{\|\tilde{y}_0^{k+1}\|_{L^2(0,1)}}, \quad k \geq 0. \end{cases}$$

The real sequence  $\{\|\tilde{y}_0^k\|_{L^2(0,1)}\}_{k>0}$  converges to the eigenvalue with largest module of the operator  $\mathcal{A}_\varepsilon$ :

$$\sqrt{\|\tilde{y}_0^k\|_{L^2(0,1)}} \rightarrow K(\varepsilon, T, M) \text{ as } k \rightarrow \infty.$$

The  $L^2$ -sequence  $\{y_0^k\}_k$  then converges toward the corresponding eigenvector.

**Remark** The first step requires to determine the control of minimal  $L^2$  for (9) with initial condition  $y_0^k$ .



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For a fixed initial data  $y^0 \in L^2(0, 1)$  and  $\varepsilon$  small, the numerical approximation of controls of minimal  $L^2$ -norm is a **serious challenge** :

- the minimization of  $J^*$  is **ill-conditioned** : the infimum  $\varphi_T$  lives in a huge dual space !!! this implies that the minimizer  $\varphi_T$  is highly oscillating at time  $T$  leading to high oscillations of the control  $\varepsilon\varphi_{,x}(0, \cdot)$ ;
- **Tychonoff like regularization**

$$\inf_{\varphi_T \in H_0^1(0,1)} J_\beta^*(\varphi_T) := J^*(\varphi_T) + \beta \|\varphi_T\|_{H_0^1(0,1)} \longrightarrow \|y^\varepsilon(\cdot, T)\|_{H^{-1}(0,1)} \leq \beta$$

is **meaningless** here for  $T \geq 1/|M|$  because the uncontrolled solution  $y^\varepsilon(\cdot, T)$  goes to zero with  $\varepsilon$ ;

- **Several singular layers** occur for  $y^\varepsilon$  and  $\varphi^\varepsilon$  and requires fine discretization and adapted meshes.

We use the variational approach developed in [[Fernandez-Cara-Münch, 2013](#)], [[De Souza-Münch, 2015](#)] leading to convergent approximation with respect to the discretization parameter ( $\varepsilon$  being fixed).

We replace the minimization over  $\varphi(\cdot, T)$  by the **minimization over  $\varphi$  subjected to the constraint equality  $L_\varepsilon^* \varphi = 0$** ; a **lagrange multiplier  $\lambda \in L^2(Q_T)$**  is introduced and then the saddle-point problem :

$$\sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi} \mathcal{L}(\varphi, \lambda) := \frac{1}{2} \|\varepsilon \rho_0^{-1} \varphi_x(0, \cdot)\|_{L^2(0, T)}^2 + (y_0, \varphi(0, \cdot))_{L^2(0, 1)} + \langle \lambda, \rho^{-1} L_\varepsilon^* \varphi \rangle_{L^2(Q_T)}$$

where  $\rho, \rho_0$  denote appropriate Carleman weights in  $L^\infty(Q_{T-\delta})$ .

**Remarks** : • a conformal approximation of  $\Phi$  leads to **strong convergent approximation** of the controls;

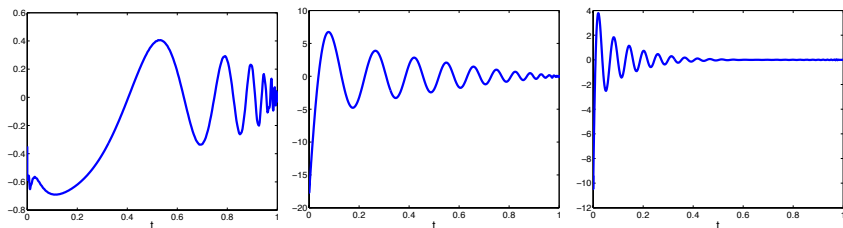
- This leads to a space-time approach well-suited for **mesh adaptivity**.

# A FreeFem++ code associated to the space-time variational formulation

```
1 border bas(s=0,1){x=s; y=0;label=Ntop;}; border droit(s=0,T){x=1;y=s;label=Nright;};
2 border haut(s=1,0){x=s;y=T;label=Nhaut;}; border gauche(s=T,0){x=0;y=s;label=Ngauche;};
3 mesh Th=buildmesh(bas(50)+droit(50)+haut(50)+gauche(50));
4
5 fespace Vh(Th,P3); fespace Ph(Th,P3);
6 real eps=1.e-3, M=1, r1=1.e-6, r2=1.e-6, alpha1=5.e-2, alpha2=5.e-2;
7
8 Vh phi,p,phit,pt; Ph l1,l2,l1t,l2t; Vh y0 = sin(pi*x)*(1-y);
9
10 problem transport([phi,p,l1,l2],[phit,pt,l1t,l2t])=
11 // Initial conjugate cost
12 int1d(Th,Ngauche) (eps*eps*dx(phi)*dx(phit))+int1d(Th,Nbas) (y0*phit)
13
14 // bilinear adjoint- direct solution terms
15 + int2d(Th) ((dy(phi)+dx(p)+M*dx(phi))*l1t)
16 + int2d(Th) ((dy(phit)+dx(pt)+M*dx(phit))*l1)
17 + int2d(Th) ((p-eps*dx(phi))*l2t)
18 + int2d(Th) ((pt-eps*dx(phit))*l2)
19
20 // Augmentation terms
21 + int2d(Th) (r1*(dy(phi)+dx(p)+M*dx(phi))* (dy(phit)+dx(pt)+M*dx(phit)))
22 + int2d(Th) (r2*(eps*dx(phi)-p) * (eps*dx(phit)-pt))
23
24 // stabilized terms
25 -int2d(Th) (alpha1*(dy(l1)+M*dx(l1)-eps*dx(l2))* (dy(l1t)+M*dx(l1t)-eps*dx(l2t)))
26 - int2d(Th) (alpha2*(dx(l1)-l2)*(dx(l1t)-l2t))
27
28 // boundary conditions for the adjoint and lagrange multiplier solutions
29 + on(Nbas,l1=y0)+on(Ndroit,Ngauche,phi=0.)+on(Ndroit, Nhaut, l1=0.);
```

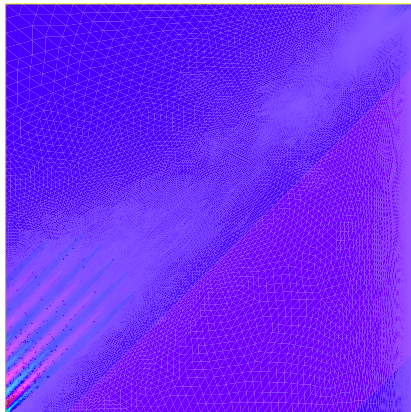
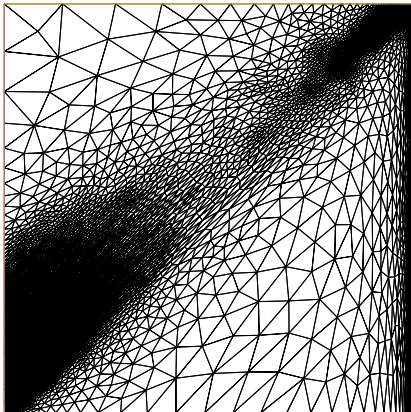
# Picture of controls with respect to $\varepsilon$ , $y_0$ fixed

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = 1$$



Control of minimal  $L^2(0, T)$ -norm  $v^\varepsilon(t) \in [0, T]$  for  $\varepsilon = 10^{-1}, 10^{-2}$  and  $10^{-3}$ .

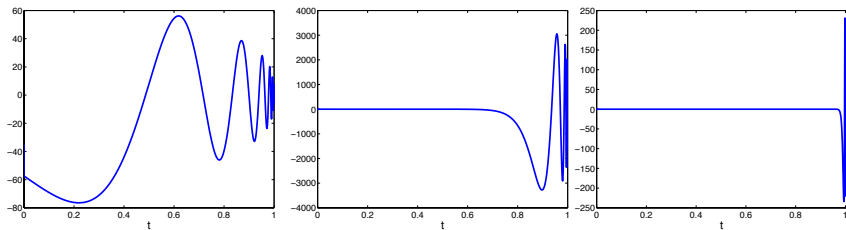
# One adapted mesh over the space-time $Q_T$



$$y_0(x) = \sin(\pi x) - M = 1 - \varepsilon = 10^{-3}.$$

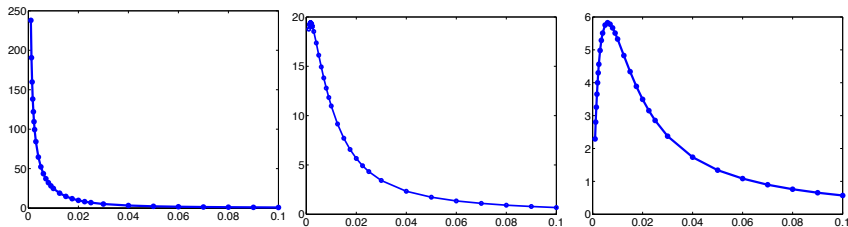
# Picture of controls with respect to $\varepsilon$ , $y_0$ fixed

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = -1$$



Control of minimal  $L^2(0, T)$ -norm  $v^\varepsilon(t) \in [0, T]$  for  $\varepsilon = 10^{-1}, 10^{-2}$  and  $10^{-3}$ .

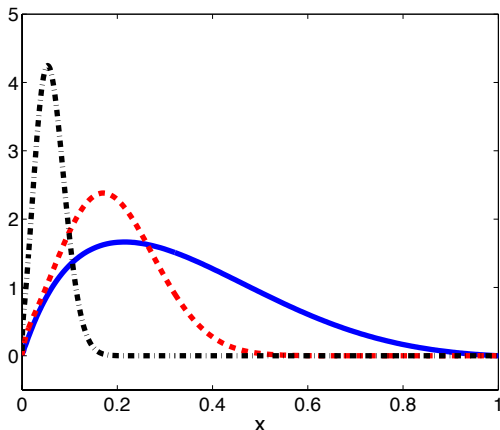
# Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon - M = 1$ .



Cost of control w.r.t.  $\varepsilon \in [10^{-3}, 10^{-1}]$  for  $T = 0.95 \frac{1}{M}$ ,  $T = 1 \frac{1}{M}$  and  $T = 1.05 \frac{1}{M}$



## Corresponding worst initial condition



$T = 1 - M = 1$  - The optimal initial condition  $y_0$  in  $(0, 1)$  for  $\varepsilon = 10^{-1}$ ,  $\varepsilon = 10^{-2}$  and  $\varepsilon = 10^{-3}$ .

$\implies y_0$  is close to  $e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x) / \|e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)\|_{L^2(0,1)}$



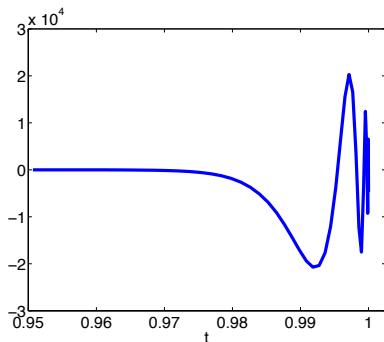
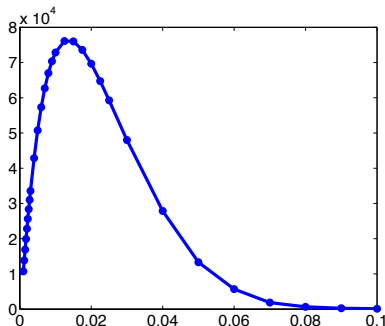
# Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon: M \in \{1, -1\}$

$\varepsilon$	$T = 1.$
$10^{-3}$	18.7555
$1.25 \times 10^{-3}$	19.1953
$1.5 \times 10^{-3}$	19.3883
$1.75 \times 10^{-3}$	19.4234
$2 \times 10^{-3}$	19.3540
$2.25 \times 10^{-3}$	19.2093
$2.5 \times 10^{-3}$	19.0163
$3 \times 10^{-3}$	18.5275
$4 \times 10^{-3}$	17.3600
$5 \times 10^{-3}$	16.1269
$6 \times 10^{-3}$	14.9392
$7 \times 10^{-3}$	13.8166
$8 \times 10^{-3}$	12.7839
$9 \times 10^{-3}$	11.8380
$10^{-2}$	10.9763
$10^{-1}$	0.6808

$\varepsilon$	$T = 1.$
$10^{-3}$	10718.0955
$1.25 \times 10^{-3}$	13839.4039
$1.5 \times 10^{-3}$	16903.9918
$1.75 \times 10^{-3}$	19898.1360
$2 \times 10^{-3}$	22812.2634
$2.25 \times 10^{-3}$	25638.7601
$2.5 \times 10^{-3}$	28375.3693
$3 \times 10^{-3}$	33575.9482
$4 \times 10^{-3}$	42871.1424
$5 \times 10^{-3}$	50751.4443
$6 \times 10^{-3}$	57316.7716
$7 \times 10^{-3}$	62692.7273
$8 \times 10^{-3}$	66997.3602
$9 \times 10^{-3}$	70350.3966
$10^{-2}$	72862.0738
$10^{-1}$	123.3069

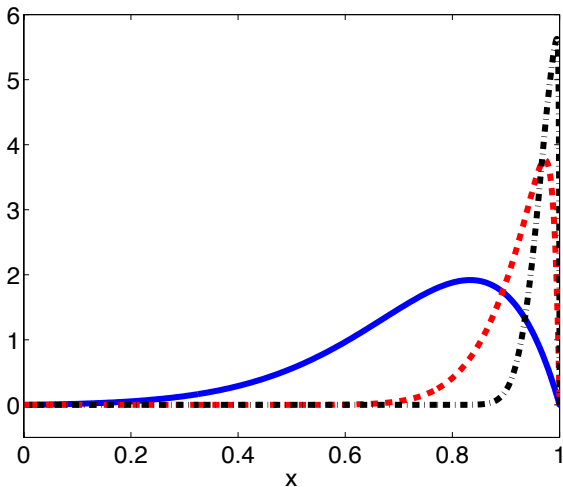
Cost  $K(\varepsilon, T, M)$  w.r.t  $\varepsilon$  for  $M = 1$ (Left) and  $M = -1$  (Right).

# Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon - M = -1$



**Left:** Cost of control w.r.t.  $\varepsilon$  for  $T = \frac{1}{|M|}$ ; **Right:** Corresponding control  $v^\varepsilon$  in the neighborhood of  $T$  for  $\varepsilon = 10^{-3}$

## Corresponding worst initial condition for $M = -1$



$T = 1 - M = -1$  - The optimal initial condition  $y_0$  in  $(0, 1)$  for  $\varepsilon = 10^{-1}$ ,  $\varepsilon = 10^{-2}$  and  $\varepsilon = 10^{-3}$ .

- $y_0^\varepsilon(x) = K_\varepsilon \sin(\pi x) \exp(-\frac{Mx}{2\varepsilon})$  is a candidate !
- Estimation of the corresponding  $L^2$  minimal control norm ?
- Weak limit of the system for  $\varepsilon$ -dependent initial condition ?
- The negative case  $M < 0$  is out of reach numerically ! (we need to wait at least  $T = 2\sqrt{2}/|M|$  for which the free solution satisfies  $\|y^\varepsilon(\cdot, T)\|_2 \approx e^{-|M|/\varepsilon}$ )

## Second Attempt : Asymptotic analysis w.r.t. $\varepsilon$ of the optimality system

We take  $M > 0$ .

Optimality system :

$$\left\{ \begin{array}{ll} L_\varepsilon y^\varepsilon = 0, & L_\varepsilon^* \varphi^\varepsilon = 0, & (x, t) \in Q_T, \\ y^\varepsilon(\cdot, 0) = y_0^\varepsilon, & & x \in (0, 1), \\ v^\varepsilon(t) = y^\varepsilon(0, t) = \varepsilon \varphi_x^\varepsilon(0, t), & & t \in (0, T), \\ y^\varepsilon(1, t) = 0, & & t \in (0, T), \\ \varphi^\varepsilon(0, t) = \varphi^\varepsilon(1, t) = 0, & & t \in (0, T), \\ -\beta(\varepsilon) \varphi_{xx}^\varepsilon(\cdot, T) + y^\varepsilon(\cdot, T) = 0, & & x \in (0, 1). \end{array} \right.$$

$\beta(\varepsilon) \geq 0$ - Regularization parameter

J.-L. Lions *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*. Lecture Notes in Mathematics. Springer 1973.

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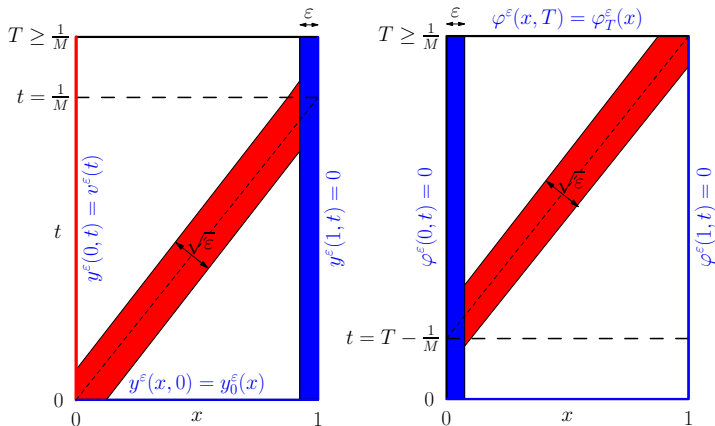
$$\left\{ \begin{array}{ll} L_\varepsilon y^\varepsilon = 0, & L_\varepsilon^* \varphi^\varepsilon = 0, & (x, t) \in Q_T, \\ y^\varepsilon(\cdot, 0) = y_0^\varepsilon, & & x \in (0, 1), \\ v^\varepsilon(t) = y^\varepsilon(0, t) = \varepsilon \varphi_x^\varepsilon(0, t), & & t \in (0, T), \\ y^\varepsilon(1, t) = 0, & & t \in (0, T), \\ \varphi^\varepsilon(0, t) = \varphi^\varepsilon(1, t) = 0, & & t \in (0, T), \\ -\beta(\varepsilon) \varphi_{xx}^\varepsilon(\cdot, T) + y^\varepsilon(\cdot, T) = 0, & & x \in (0, 1). \end{array} \right.$$

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The situation is tricky because (assume  $M > 0$ )

- $y^\varepsilon$  exhibits a boundary layer of size  $\mathcal{O}(\varepsilon)$  at  $x = 1$  and an inner layer of size  $\mathcal{O}(\sqrt{\varepsilon})$  along the characteristic  $\{(x, t) \in Q_T, x - Mt = 0\}$ ;
- $\varphi^\varepsilon$  exhibits a boundary layer of size  $(\mathcal{O}(\varepsilon))$  at  $x = 0$  and an inner layer of size  $(\mathcal{O}(\sqrt{\varepsilon}))$  along the characteristic  $\{(x, t) \in Q_T, x - M(t - T) - 1 = 0\}$ ;



Singular layers zone for  $y^\varepsilon$  (left) and  $\varphi^\varepsilon$  (right) in the case  $M > 0$ .



# Direct problem - Matched asymptotic expansion method - With compatibility conditions

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x, t) \in (0, 1) \times (0, T), \\ y^\varepsilon(0, t) = v^\varepsilon(t) = v(t), \quad y^\varepsilon(1, t) = 0, & t \in (0, T), \\ y^\varepsilon(x, 0) = y_0(x), & x \in (0, 1), \end{cases} \quad (5)$$

$v$  being known.

We construct an **asymptotic approximation** of the solution  $y^\varepsilon$  of (5) by using **the matched asymptotic expansion method** [Cole96, Echkauss81]. We consider two formal asymptotic expansions of  $y^\varepsilon$ :

– the **outer expansion**

$$\sum_{k=0}^m \varepsilon^k y^k(x, t), \quad (x, t) \in (0, T)$$

– the **inner expansion** (boundary layer at  $x = 1$ )

$$\sum_{k=0}^m \varepsilon^k Y^k(z, t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}), \quad t \in (0, T)$$

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– the **outer expansion**

$$\sum_{k=0}^m \varepsilon^k y^k(x, t), \quad (x, t) \in (0, T) \implies y_t^k + M y_x^k = y_{xx}^{k-1}$$

– the **inner expansion** (boundary layer at  $x = 1$ )

$$\sum_{k=0}^m \varepsilon^k Y^k(z, t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}), \quad t \in (0, T) \implies Y_{zz}^k + M Y_z^k = Y_t^{k-1}$$

# Direct problem - Outer expansion - $y^k$ - Case 1

$$y^0(x, t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any  $1 \leq k \leq m$ ,

$$y^k(x, t) = \begin{cases} \int_0^t y_{xx}^{k-1}(x + (s-t)M, s) ds, & x > Mt, \\ \int_0^{x/M} y_{xx}^{k-1}(sM, t - \frac{x}{M} + s) ds, & x < Mt. \end{cases}$$

For instance,

$$y^1(x, t) = \begin{cases} t y_0''(x - Mt), & x > Mt, \\ \frac{x}{M^3} v''\left(t - \frac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x, t) = \begin{cases} \frac{t^2}{2} y_0^{(4)}(x - Mt), & x > Mt, \\ -\frac{2x}{M^5} (v^{(3)})\left(t - \frac{x}{M}\right) + \frac{x^2}{2M^6} v^{(4)}\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

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## Lemma

$$Y^0(z, t) = y^0(1, t) \left(1 - e^{-Mz}\right), \quad (z, t) \in (0, +\infty) \times (0, T).$$

For any  $1 \leq k \leq m$ , the solution reads

$$Y^k(z, t) = Q^k(z, t) + e^{-Mz} P^k(z, t), \quad (z, t) \in (0, +\infty) \times (0, T),$$

where

$$P^k(z, t) = - \sum_{i=0}^k \frac{1}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1, t) z^i, \quad Q^k(z, t) = \sum_{i=0}^k \frac{(-1)^i}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1, t) z^i.$$

$\chi_\varepsilon$  - a cut-off function

### Theorem (Amirat, M)

Let  $\gamma \in (0, 1)$ . Let  $y^\varepsilon$  be the solution of problem (5) and let  $w_0^\varepsilon$  be the function defined as follows

$$w_0^\varepsilon(x, t) = \chi_\varepsilon(x)y^0(x, t) + (1 - \chi_\varepsilon(x))Y^0\left(\frac{1-x}{\varepsilon}, t\right).$$

Assume that  $y_0 \in H^2(0, 1)$ ,  $v^0 \in H^2(0, T)$ , and the *matching conditions*

$$v^0(t=0) = y_0(x=0), \quad M(v^0)'(t=0) + y_0'(x=0) = 0. \quad (6)$$

Then there is a constant  $c_0$  *independent of  $\varepsilon$*  such that

$$\|y^\varepsilon - w_0^\varepsilon\|_{C([0, T]; L^2(0, 1))} \leq c_0 \varepsilon^{\frac{1}{2}\gamma}.$$

## Theorem (Amirat, M, 19)

Let  $y^\varepsilon$  be the solution of problem (5) and let  $w_m^\varepsilon$  be the function defined as follows

$$w_m^\varepsilon(x, t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^m \varepsilon^k y^k(x, t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^m \varepsilon^k Y^k \left( \frac{1-x}{\varepsilon}, t \right). \quad (7)$$

Assume that  $y_0 \in C^{2m+1}[0, 1]$ ,  $v \in C^{2m+1}[0, T]$ , and that the  $C^{2m+1}$ -*matching conditions* are satisfied

$$v^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p+1} y_0}{\partial x^{i+2} \partial t^j}(0, 0), \quad 0 \leq p \leq 2m+1.$$

Then there is a constant  $c_m$  *independent of  $\varepsilon$*  such that (for any  $\gamma \in (0, 1/2]$ )

$$\|y^\varepsilon(\cdot, t) - w_m^\varepsilon(\cdot, t)\|_{L^2(0,1)} \leq c_m \varepsilon^{\frac{2m+1}{2}\gamma} + c_m \left( \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{(2m+3)\gamma}{2}} \right) e^{-\frac{M^2}{2\varepsilon}\gamma t}, \quad \forall t \in [0, T].$$

## Proposition

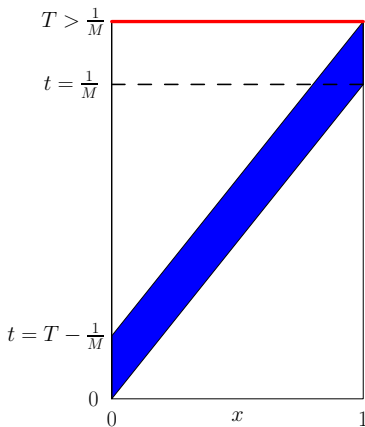
Let  $m \in \mathbb{N}$ ,  $T > \frac{1}{M}$  and  $a \in ]0, T - \frac{1}{M}[$ . Assume regularity and matching conditions on the initial condition  $y_0$  and function  $v$ . Assume moreover that

$$v(t) = 0, \quad \forall t \in [a, T].$$

Then, the solution  $y^\varepsilon$  of problem (5) satisfies the following property

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c_m \varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall \gamma \in (0, 1)$$

for some constant  $c_m > 0$  independent of  $\varepsilon$ .  
The function  $v \in C([0, T])$  is an **approximate null control** for (9).





# Convergence w.r.t $m$ under conditions on $y_0$ and the $v$ .

(i) The initial condition  $y_0$  belongs to  $C^\infty[0, 1]$  and there is  $b \in \mathbb{R}$  such that

$$\|y_0^{(k)}\|_{L^2(0,1)} \leq \left\lfloor \frac{k}{2} \right\rfloor! b^{\frac{k}{2}}, \quad \forall k \in \mathbb{N}.$$

(ii)  $v$  is a polynomial of degree  $\leq p - 1$ ,  $p \geq 1$ , uniformly bounded in  $C^{p-1}[0, T]$ .

(iii) For any  $m \in \mathbb{N}$ , the functions  $v$  and  $y_0$  satisfy the matching conditions.

## Theorem

Assume (i)-(ii)-(iii).  $\exists \varepsilon_0 > 0$  such that for any fixed  $0 < \varepsilon < \varepsilon_0$ , we have

$$\begin{aligned} \mathcal{X}_\varepsilon(x) \sum_{k=0}^{\infty} \varepsilon^k y^k(x, t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{\infty} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right) + \tilde{\theta}^\varepsilon(x) \\ = y^\varepsilon(x, t) \quad \text{a.e. in } Q_T. \end{aligned}$$

The corrector function  $\tilde{\theta}^\varepsilon \in L^2(H_0^1) \cap C(L^2)$  solves

$$\begin{cases} L_\varepsilon(\tilde{\theta}^\varepsilon) = f^\varepsilon, & (x, t) \in Q_T, \\ \tilde{\theta}^\varepsilon(0, t) = \tilde{\theta}^\varepsilon(1, t) = 0, & t \in (0, T), \\ \tilde{\theta}^\varepsilon(x, 0) = (1 - \mathcal{X}_\varepsilon(x)) e^{-M\frac{1-x}{\varepsilon}} \sum_{i=0}^{\infty} \frac{y_0^{(i)}(1)}{i!} (1-x)^i, & x \in (0, 1), \end{cases} \quad x \in (0, 1).$$

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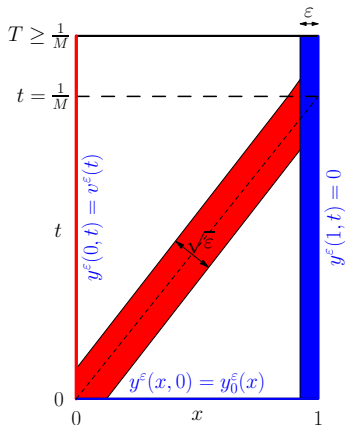
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Singular layers zone for  $y^\epsilon$  in the case  $M > 0$ .

Occurrence of two interacting singular layers of different sizes !

Very few papers dealing with the asymptotic analysis of PDEs involving two interacting singular layers :

- [W. Eckhaus, W. and E.M. de Jager](#), *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type*, Arch. Rational Mech. Anal., 1966.

$$\begin{cases} u_x^\varepsilon(x, y) - \varepsilon \Delta u^\varepsilon(x, y) = 0, & (x, y) \in (0, 1) \times (-1, 1), \\ u^\varepsilon(0, y) = f(y), \quad u^\varepsilon(x, -1) = u^\varepsilon(x, 1) = u^\varepsilon(1, y) = 0, & x \in [0, 1], y \in [-1, 1], \end{cases} \quad (8)$$

where  $f : [-1, 1] \rightarrow \mathbb{R}$  is a piecewise constant, discontinuous at  $y = 0$ .

- [Larry Bobisud](#), *Second-order linear parabolic equations with a small parameter*, Arch. Rational Mech. Anal., 1967.

$$\begin{cases} u_t^\varepsilon - \varepsilon u_{xx}^\varepsilon + u_x^\varepsilon + u^\varepsilon = 0, & \text{in } Q_T, \\ u^\varepsilon(0, \cdot) = f, \quad u^\varepsilon(1, \cdot) = g, & \text{in } (0, T), \\ u^\varepsilon(\cdot, 0) = u_0, & \text{in } (0, 1). \end{cases} \quad (9)$$

Assuming  $u_0 \in C^4([0, 1])$  and  $f, g \in C^3([0, T])$ , obtention of  $w^\varepsilon$  such that  $\|u^\varepsilon - w^\varepsilon\|_{L^\infty(Q_T)} = \mathcal{O}(\sqrt{\varepsilon})$  by the way of **a maximum principle**.

We now take into account the boundary layer on the characteristic and consider three formal asymptotic expansions of  $y^\varepsilon$ :

– the outer expansion

$$\sum_{k=0}^m \varepsilon^k y^k(x, t), \quad (x, t) \in Q_T, \quad x - Mt \neq 0$$

– the first inner expansion (on the characteristic  $x - Mt = 0$ )

$$\sum_{k=0}^m \varepsilon^{\frac{k}{2}} W^{k/2}(w, t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}, \quad t \in (0, T)$$

– the second inner expansion (at  $x = 1$ )

$$\sum_{k=0}^m \varepsilon^{k/2} Y^{k/2}(z, \tau, t), \quad z = \frac{1 - x}{\varepsilon}, \quad \tau = \frac{1 - t}{\sqrt{\varepsilon}}$$

# Direct problem - Matched asymptotic expansion method

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$$\sum_{k=0}^m \varepsilon^k y^k(x, t), \quad (x, t) \in Q_T, \quad x - Mt \neq 0 \implies y_t^k + M y_x^k = y_{xx}^{k-1}$$

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$$\sum_{k=0}^m \varepsilon^{\frac{k}{2}} W^{k/2}(w, t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}, \quad t \in (0, T) \implies W_t^{k/2} - W_{ww}^{k/2} = 0$$

– the second inner expansion (at  $x = 1$ )

$$\sum_{k=0}^m \varepsilon^{k/2} Y^{k/2}(z, \tau, t), \quad z = \frac{1 - x}{\varepsilon}, \quad \tau = \frac{1 - Mt}{\sqrt{\varepsilon}} \implies Y_{zz}^{k/2} + M Y_z^{k/2} = Y_t^{(k-2)/2} - Y_\tau^{(k-1)/2}$$



## Example: the first term

- -  $y^0$  solves the transport eq.:

$$\begin{cases} y_t^0 + My_x^0 = 0, & (x, t) \in Q_T \\ y^0(x, 0) = y_0, y^0(0, t) = v^0 \end{cases} \implies y^0(x, t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v^0\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

- -  $W^0$  solves the heat eq.:

$$\begin{cases} W_t^0(w, t) - W_{ww}^0(w, t) = 0, & (w, t) \in \mathbb{R} \times (0, T), \\ \lim_{w \rightarrow +\infty} W^0(w, t) = y^0((Mt)^+, t) = y_0(0), & t \in (0, T), \\ \lim_{w \rightarrow -\infty} W^0(w, t) = y^0((Mt)^-, t) = v(t), & t \in (0, T). \end{cases}$$

$$W^0(w, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(w-s)^2}{4t}} g_0(s) ds, \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}$$
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## Example: the first term

- $p_\varepsilon^0(x, t) = y^0(x, t) + W^0(w, t) - y^0((Mt)^\pm, t)$

$$p_\varepsilon^0(x, t) = \begin{cases} y_0(x - Mt) + W_\varepsilon^0(w, t) - y_0(0), & x > Mt, \\ v\left(t - \frac{x}{M}\right) + W_\varepsilon^0(w, t) - v(0), & x \leq Mt. \end{cases} \quad (10)$$

- $Y^0$  solves the ODE:  $z = \frac{1-x}{\varepsilon}, \tau = \frac{1/M-t}{\sqrt{\varepsilon}}$

$$\begin{cases} Y_{zz}^0(z, \tau, t) + MY_z^0(z, \tau, t) = 0, & (z, \tau, t) \in \mathbb{R}_*^+ \times \mathbb{R} \times (0, T), \\ Y^0(0, \tau, t) = 0, \quad \lim_{z \rightarrow +\infty} Y^0(z, \tau, t) = p_\varepsilon^0(1, t), & (\tau, t) \in \mathbb{R} \times (0, T). \end{cases}$$

$$Y^0(z, \tau, t) = p_\varepsilon^0(1, t) \left(1 - e^{-Mz}\right), \quad (z, \tau, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T).$$

- $P_\varepsilon^0(x, t) = p_\varepsilon^0(x, t) + Y^0(z, \tau, t) - p_\varepsilon^0(1, t) = p_\varepsilon^0(x, t) - p_\varepsilon^0(1, t)e^{-Mz}$

## Example: the first term

- $p_\varepsilon^0(x, t) = y^0(x, t) + W^0(w, t) - y^0((Mt)^\pm, t)$

$$p_\varepsilon^0(x, t) = \begin{cases} y_0(x - Mt) + W_\varepsilon^0(w, t) - y_0(0), & x > Mt, \\ v\left(t - \frac{x}{M}\right) + W_\varepsilon^0(w, t) - v(0), & x \leq Mt. \end{cases} \quad (10)$$

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# Direct problem - Matched asymptotic expansion method - Case 2 - First order approximation (1)

Let

$$P^\varepsilon = P_\varepsilon^0 + \sqrt{\varepsilon}P_\varepsilon^{1/2} + \varepsilon P_\varepsilon^1 + \varepsilon^{3/2}P_\varepsilon^{3/2}$$

## Theorem (Amirat, M '19)

Assume  $v \in H^3([0, T])$ ,  $y_0 \in H^3([0, 1])$ . Then  $\exists C > 0$  independent of  $\varepsilon$  s.t.

$$\left\| y^\varepsilon(\cdot, t) - P^\varepsilon(\cdot, t) \right\|_{L^2(0,1)} \leq C(\varepsilon^{3/2} + \varepsilon^{1/2} e^{-\frac{M^2}{2\varepsilon^{1/2}} t}) \quad \forall t \in [0, T]$$

and (assuming  $y_0(1) = y_0'(1) = 0$ )

$$\|(y^\varepsilon - P^\varepsilon)_x\|_{L^2(Q_T)} \leq C\varepsilon$$

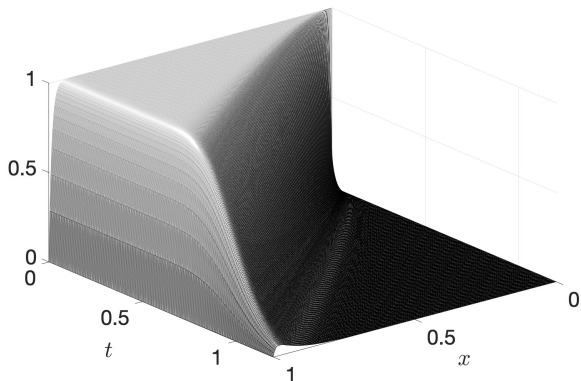
As an illustration, we consider the simple case  $v \equiv 0$  and  $y_0 \equiv 1$  for which

$$\left\{ \begin{array}{l} P^\varepsilon(x, t) = W_\varepsilon^0(w, t) - \left( W_\varepsilon^0(M\tau, t) + \varepsilon^{1/2} z W_{\varepsilon, w}^0(M\tau, t) + \right. \\ \left. \varepsilon \frac{z^2}{2} W_{\varepsilon, ww}^0(M\tau, t) + \varepsilon^{3/2} \frac{z^3}{6} W_{\varepsilon, www}^0(M\tau, t) \right) e^{-Mz}, \\ w = \frac{x - Mt}{\sqrt{\varepsilon}}, \quad M\tau = \frac{1 - Mt}{\sqrt{\varepsilon}}, \quad z = \frac{1 - x}{\varepsilon}. \end{array} \right.$$

with

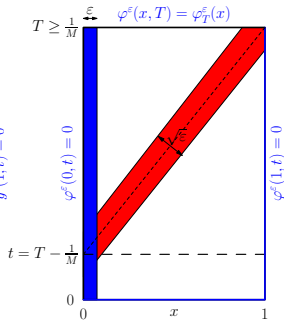
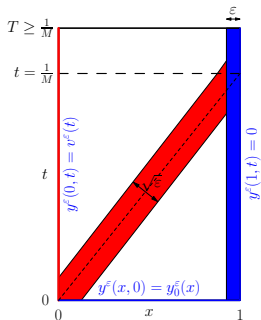
$$\begin{aligned} W_\varepsilon^0(w, t) = & \frac{y_0(0) - v(0)}{2} \operatorname{erf}\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0(0) + v(0)}{2} \\ & + \frac{v(0) - y_0(0)}{2} e^{\frac{Mw}{\sqrt{\varepsilon}} + \frac{M^2 t}{\varepsilon}} \operatorname{erfc}\left(\frac{w}{2\sqrt{t}} + \frac{M\sqrt{t}}{\sqrt{\varepsilon}}\right) \end{aligned} \quad (11)$$





$P^\varepsilon$  in  $(0, 1) \times (0, 1.2/M)$ ;  $M = 1$ ,  $\varepsilon = 10^{-2}$ ;  $v \equiv 0$ ,  $y_0 \equiv 1$ .

$$\left\{ \begin{array}{ll} L_\varepsilon y^\varepsilon = 0, & L_\varepsilon^* \varphi^\varepsilon = 0, & (x, t) \in Q_T, \\ y^\varepsilon(\cdot, 0) = y_0^\varepsilon, & & x \in (0, 1), \\ v^\varepsilon(t) = y^\varepsilon(0, t) = \varepsilon \varphi_x^\varepsilon(0, t), & & t \in (0, T), \\ y^\varepsilon(1, t) = 0, & & t \in (0, T), \\ \varphi^\varepsilon(0, t) = \varphi^\varepsilon(1, t) = 0, & & t \in (0, T), \\ -\beta(\varepsilon) \varphi_{xx}^\varepsilon(\cdot, T) + y^\varepsilon(\cdot, T) = 0, & & x \in (0, 1), \\ \beta(\varepsilon) = \mathcal{O}(\varepsilon^m) & & \end{array} \right. \quad (12)$$



$$y_0^\varepsilon(x) = c_\varepsilon e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)$$

We introduce the following change of unknown

$$y^\varepsilon(x, t) = c_\varepsilon e^{-\frac{Mx}{2\varepsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} z^\varepsilon(x, t), \quad \forall (x, t) \in Q_T,$$

leading to

$$L_\varepsilon y^\varepsilon := c_\varepsilon e^{-\frac{Mx}{2\varepsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} \left( z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + 2Mz_x^\varepsilon - \frac{M^2}{4\varepsilon} (\gamma + 3) z^\varepsilon \right).$$

$$\begin{cases} y_{tt}^\varepsilon + \varepsilon \Delta^2 y^\varepsilon - \Delta y^\varepsilon = 0, & \text{in } Q_T, \\ y^\varepsilon = 0, \quad \partial_\nu y^\varepsilon = v^\varepsilon 1_{\Gamma_T}, & \text{on } \Sigma_T, \\ (y^\varepsilon(\cdot, 0), y_t^\varepsilon(\cdot, 0)) = (y_0, y_1), & \text{on } \Omega. \end{cases}$$

## Theorem (Lions 86)

Assume  $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$  and that  $(\Omega, \Gamma_T, T)$  satisfies a geometric control condition. For any  $\varepsilon > 0$ , let  $v^\varepsilon$  be the control of minimal  $L^2(\Gamma_T)$ -norm. Then,

$$(-\sqrt{\varepsilon} v^\varepsilon, y^\varepsilon) \rightharpoonup (v, y) \text{ in } L^2(\Gamma_T) \times L^\infty(0, T; L^2(\Omega)), \text{ as } \varepsilon \rightarrow 0$$

where  $v$  is the control of minimal  $L^2(\Gamma_T)$ -norm for  $y \in C(L^2) \cap C^1(H^{-1})$ , solution of :

$$\begin{cases} y_{tt} - \Delta y = 0, & \text{in } Q_T, \\ y = v 1_{\Gamma_T}, & \text{on } \Sigma_T, \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & \text{in } \Omega. \end{cases}$$

Assuming regular initial condition, the asymptotic analysis of  $\sqrt{\varepsilon} v^\varepsilon$  is easier (and doable !) since **controls  $v^\varepsilon$  and  $v$  of minimal  $L^2$ -norm are actually more regular !** (joint work with Carlos Castro).

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THANK YOU FOR YOUR ATTENTION