Fractional Hypocoercivity

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Qualitative Behaviour of Kinetic Equations and Related Problems
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- III) Hypocoercivity
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Introduction

- Linear kinetic equations

$$\partial_t f + v \cdot \nabla_x f = \mathrm{L} f$$

- Heavy tailed local equilibrium
  - $F(v) \simeq \langle v \rangle^{-(d+\gamma)}$ with $\gamma > 0$
- Friction force
  - $E(v) \simeq \langle v \rangle^{\beta} v$ with $\beta < \gamma$

Collision operators

- Fokker-Planck operator

$$\mathrm{L}_1 f = \nabla_v \cdot \left(F\, \nabla_v (f F^{-1})\right) = \Delta_v f + \nabla_v \cdot (Ef)$$

  - With $\beta = -2$

- Scattering collision operator (or Linear Boltzmann)

$$\mathrm{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(v, v')(f' F - f F')\mathrm{d}v' = K(f) - C\langle v\rangle^{\beta} f$$

- Fractional Fokker-Planck operator

$$\mathrm{L}_3 f = \Delta_v^{\boldsymbol{a}/2} f + \nabla_v \cdot (Ef)$$

  - With $\gamma = \boldsymbol{a} + \beta$

Fractional diffusive limit

- Rescaling

$$\varepsilon^{\alpha}\,\partial_t f + \varepsilon\, v \cdot \nabla_x f = \mathrm{L} f$$

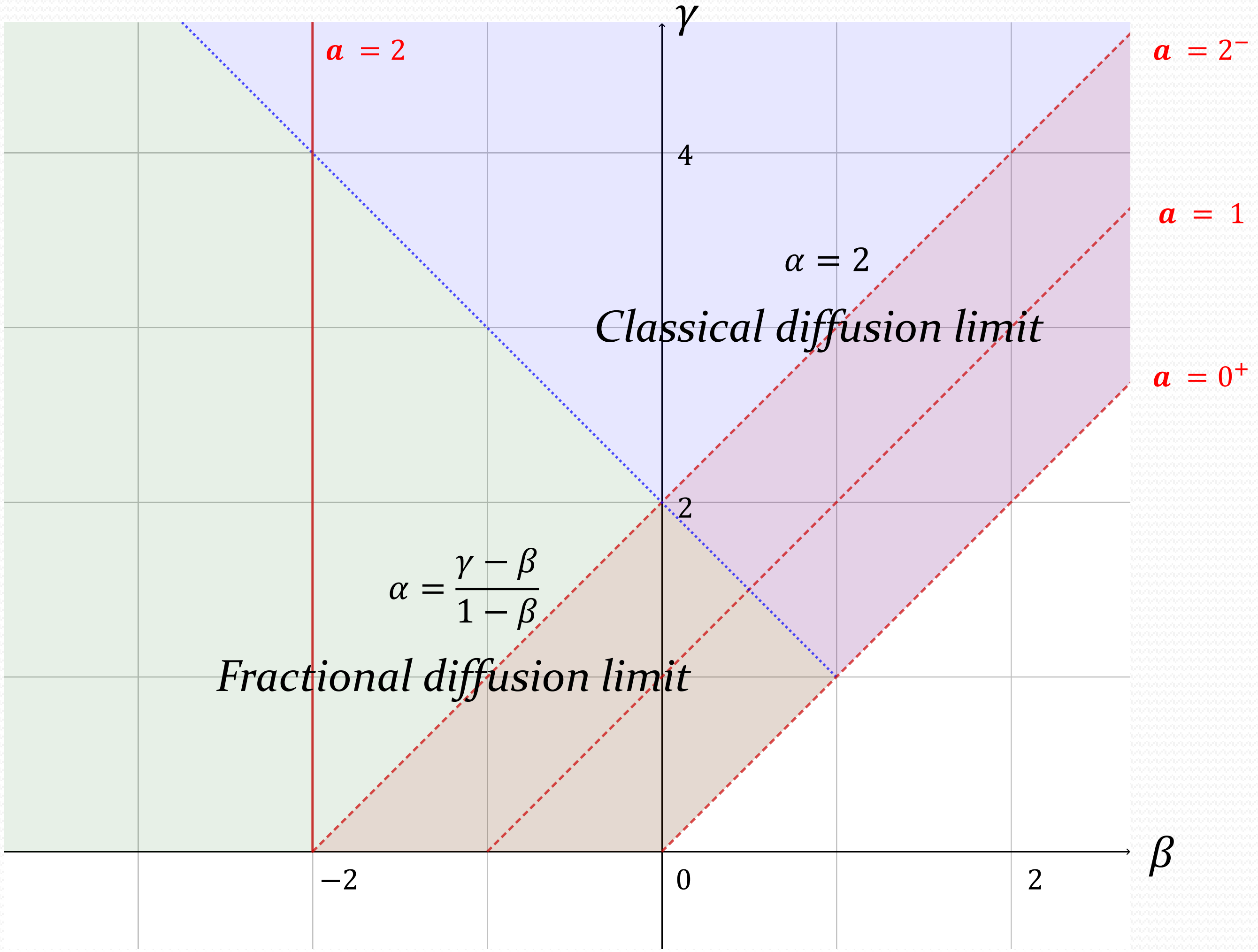
  - with $\alpha = \frac{\gamma-\beta}{1-\beta}$ if $\gamma + \beta < 2$ and $\alpha = 2$ if $\gamma + \beta > 2$

- Macroscopic limit

$$\partial_t \rho = \kappa\, \Delta^{\alpha/2} \rho$$

  - $\mathrm{L} = \mathrm{L}_1$ Degond et al. '00 ($\alpha = 2$), Mellet et al. '11
  - $\mathrm{L} = \mathrm{L}_2$ Lebeau, Puel '17 ($d = 1$), Fournier, Tardif '18
  - $\mathrm{L} = \mathrm{L}_3$ Aceves-Sanchez, Cesbron '18 ($\beta = 0$)

Fractional diffusive limit



Macroscopic asymptotic behavior

- Fractional Nash's inequality

$$\|\rho\|_{L^2(\mathrm{d}x)} \leq C \, \|\rho\|_{L^1(\mathrm{d}x)}^{\frac{\alpha}{d+\alpha}} \, \left\|\nabla^{\alpha/2}\rho\right\|_{L^2(\mathrm{d}x)}^{\frac{d}{d+\alpha}}$$

- Power law decay

$$\|\rho_t\|_{L^2(\mathrm{d}x)}^2 \lesssim \|\rho_0\|_{L^1 \cap L^2(\mathrm{d}x)}^2 \, \langle t \rangle^{-\, d/\alpha}$$

Microscopic asymptotic behavior

- If $\beta > -\gamma$, weighted Poincaré inequality
  - $\int_{\mathbb{R}^d} |h|^2 \langle v \rangle^\beta \, F \leq \int_{\mathbb{R}^d} |\nabla h|^2 F$ if $\int_{\mathbb{R}^d} hF = 0$

- Decay to local equilibrium
  - If $\beta \geq 0$

    $$\|f_t - F\|^2_{L^2(\mathrm{d}\mu)} \lesssim e^{-\lambda t} \|f_0 - F\|^2_{L^2(\mathrm{d}\mu)}$$
  - If $\beta \in (-\gamma, 0)$

    $$\|f_t - F\|^2_{L^2(\mathrm{d}\mu)} \lesssim \langle t \rangle^{-\frac{k}{|\beta|}} \|f_0 - F\|^2_{L^2(\langle v \rangle^k \mathrm{d}\mu)}$$
  - With $k \in (0, \gamma)$ and $\mathrm{d}\mu = F^{-1} \mathrm{d}v$

Hypocoercivity

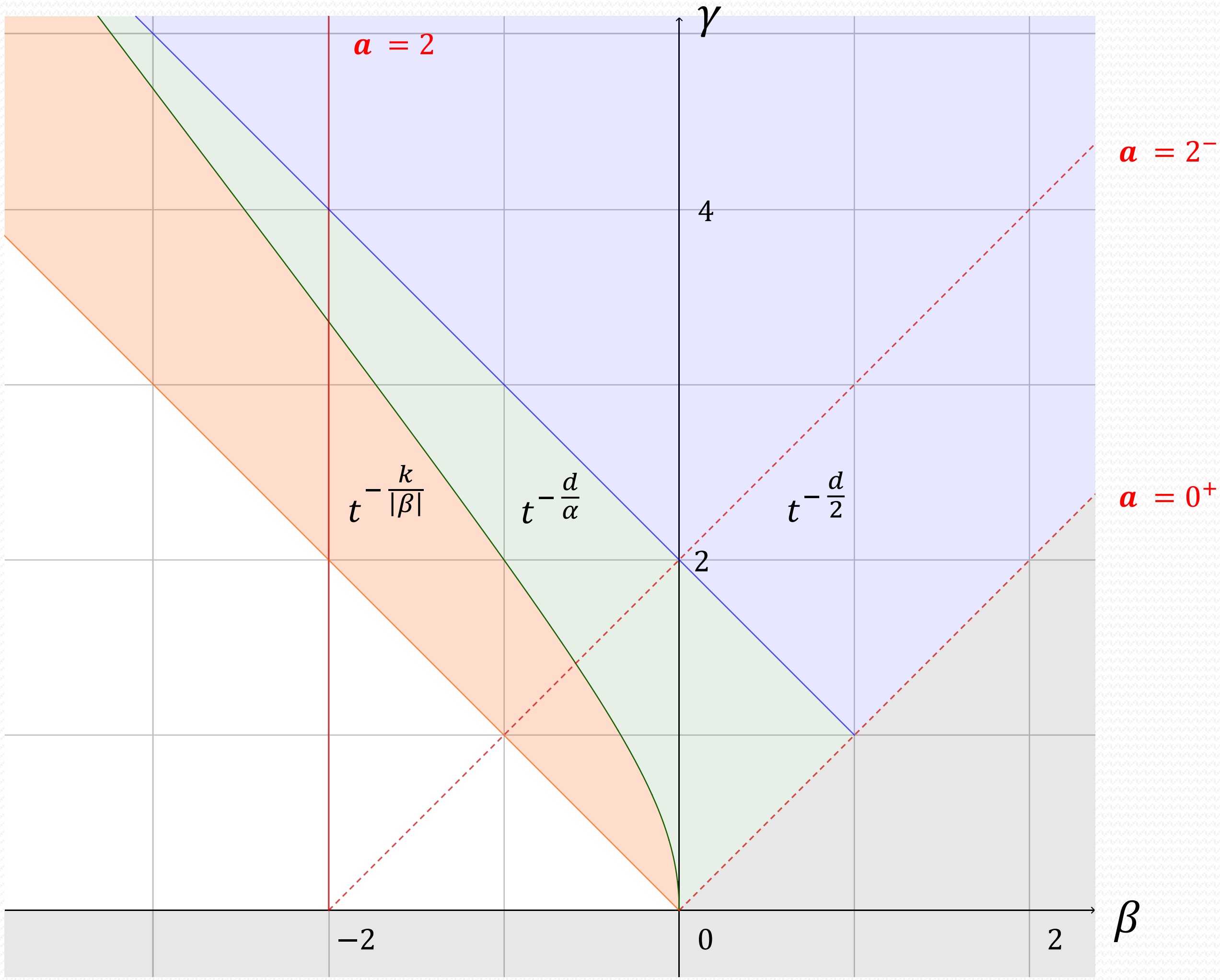
- Main Results
  - If $\beta \in (0,\gamma)$

$$\|f_t\|^2_{L^2(\mathrm{d}x\mathrm{d}\mu)} \lesssim \|f_0\|^2_{L^1(\mathrm{d}x\mathrm{d}v)\,\cap\,L^2(\mathrm{d}x\mathrm{d}\mu)}\,\langle t\rangle^{-\frac{d}{\alpha}}$$

  - If $\beta \in (-\gamma, 0)$

$$\|f_t\|^2_{L^2(\mathrm{d}x\mathrm{d}\mu)} \lesssim \|f_0\|^2_{L^1(\mathrm{d}x\mathrm{d}v)\,\cap\,L^2(\langle v\rangle^k\mathrm{d}x\mathrm{d}\mu)}\,\langle t\rangle^{-\min\left(\frac{d}{\alpha},\frac{k}{|\beta|}\right)}$$

Main results



Strategy

- Mode by mode estimates
  - Fourier transform in the x variable $\hat{f} = \hat{f}(v, \xi)$
  - $\partial_t \hat{f} + \mathrm{T}\hat{f} = \mathrm{L}\hat{f}$ with $\mathrm{T} = i\, v \cdot \xi$
  - Entropy for each fixed mode ξ for $\delta \in (0,1)$

$$\mathrm{H}_\xi(f) \coloneqq \|\hat{f}\|^2_{L^2(\mathrm{d}\mu)} + \delta \left\langle \mathrm{A}_\xi \hat{f}, \hat{f} \right\rangle_{L^2(\mathrm{d}\mu)}$$

  - Global entropy

$$\mathrm{H}(f) \coloneqq \int_{\mathbb{R}^d} \mathrm{H}_\xi(f)\, \mathrm{d}\xi$$

- Classical case (Dolbeault et al. ′15, Bouin et al. ′17)

$$\mathrm{A} = (1 + |\mathrm{T}\Pi|^2)^{-1} (\mathrm{T}\Pi)^*$$

  - With $\Pi f = F(v) \int_{\mathbb{R}^d} f \, \mathrm{d}v$

Entropy with fractional scaling

- New operator A

$$\mathrm{A}_\xi := \psi\,(\mathrm{T\Pi})^*\,\varphi_\beta$$

  - With

$$\varphi_b(\xi,v) \;:=\; \frac{\langle v\rangle^{-b}}{1+\langle v\rangle^{2|1-b|}|\xi|^2}$$

$$\psi(\xi,v) \;:=\; \varphi_0/\|\varphi_0\|_{L^2(\mathrm{d}v)}$$

- Mix between the classical entropy where
  - $\mathrm{A}_\xi = (\mathrm{T\Pi})^*\,\varphi_0$
- And the symbol appearing in Mellet et al. ′11 for the proof of the fractional diffusion limit
  - $a(\xi) = \frac{\langle v\rangle^{\beta}}{\langle v\rangle^{\beta} - i\,v\cdot\xi} = \frac{\langle v\rangle^{-\beta}(1+i\,v\cdot\xi\,\langle v\rangle^{-\beta})}{1+\langle v\rangle^{-\beta}|v\cdot\xi|^2}$

Steps of the proof

- Prove propagation of weighted Lebesgue norms: $e^{t\mathrm{L}}$ bounded in $L^2\left(\langle v\rangle^k \mathrm{d}\mu\right)$
  - $\left\|F^{-1}e^{t\mathrm{L}}\right\|_{L^1\left(F\langle v\rangle^k \mathrm{d}v\right)\to L^1\left(F\langle v\rangle^k \mathrm{d}v\right)} \leq 1$
  - $\left\|F^{-1}e^{t\mathrm{L}}\right\|_{L^\infty(\mathrm{d}v)\to L^\infty(\mathrm{d}v)} \leq 1$

- Estimate the derivative of the entropy $\quad (\mathrm{g} \coloneqq (1-\Pi)\hat{f})$

$$\frac{\mathrm{dH}_\xi}{\mathrm{d}t} = -\left\langle -\mathrm{L}\hat{f}, \hat{f}\right\rangle - \boldsymbol{\delta}\left\langle A_\xi \mathrm{T}\Pi\hat{f}, \Pi\hat{f}\right\rangle - \boldsymbol{\delta}\left\langle A_\xi \mathrm{T}\Pi\hat{f}, (1-\Pi)\hat{f}\right\rangle$$
$$+\boldsymbol{\delta}\left\langle A_\xi(1-\Pi)\hat{f}, (\mathrm{L}-\mathrm{T})(1-\Pi)\hat{f}\right\rangle + \boldsymbol{\delta}\left\langle A_\xi(\mathrm{L}-\mathrm{T})(1-\Pi)\hat{f}, \hat{f}\right\rangle$$

  - Weighted Poincaré inequality + bounded micro-macro terms!

$$\frac{\mathrm{dH}_\xi}{\mathrm{d}t} \lesssim -\delta\left(\left\|(1-\Pi)\hat{f}\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)} - \frac{|\xi|^\alpha}{\langle\xi\rangle^\alpha}\left\|\Pi\hat{f}\right\|^2_{L^2(\mathrm{d}\mu)}\right)$$

  - Nash's inequality and interpolation of weighted spaces

$$\mathrm{H}'(\mathrm{t}) \lesssim -\delta\, H(t)^{1+\frac{1}{\tau}}$$

Thank you for your attention !