Numerical approximation of the decay rate for some dissipative systems

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Motivation: the damped 1-d wave equation

Consider the wave equation with a damping term $a(x) \geq 0$,

$$\begin{cases} u_{tt} - u_{xx} + a(x)u_t = 0 & \text{for } x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = u^0(x) & \text{for } x \in (0,1) \\ u'(x,0) = u^1(x) & \text{for } x \in (0,1) \end{cases} \tag{1}$$

It is easy to see that the energy

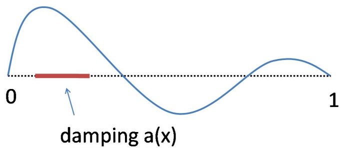
$$E(t) = \int_0^1 |u_t(x,t)|^2 dx + \int_0^1 |u_x(x,t)|^2 dx$$

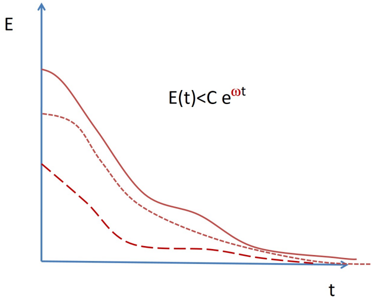
decays in time. In fact, under some conditions on $a(x)$, there exist constants C, ω such that

$$E(t) \leq Ce^{\omega t} E(0), \qquad \text{for all solutions.}$$

The optimal value of $\omega = \omega(a)$ is known as the decay rate.

Motivation: the damped 1-d wave equation





The decay rate is

$$\omega(a) = \inf\{\omega : \exists C(\omega) > 0 \text{ s.t. } \|u(t)\| \leq C\|u(0)\|e^{\omega t},$$

for every finite energy solution.

Main question: How we can recover numerically this decay rate?

Motivation: the damped 1-d wave equation

An equivalent system representation

$$\begin{cases} y' = Ay + By, & t \geq 0, \\ y(0) = y_0 \in H, \end{cases}$$

where $H = H_0^1 \times L^2(0,1)$, $A : D(A) \subset H \to H$ skewadjoint, $B : H \to H$ bounded

$$y = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -a(x) \end{pmatrix},$$

Therefore $E(t) = \|y(t)\|_H^2$ and we are interested in approximating the best ω such that

$$\|y(t)\|_H^2 \leq Ce^{\omega t}\|y(0)\|_H^2, \quad \text{for all solutions.}$$

A natural characterization of this decay rate is through the spectrum of the underlying operator $A + B$. In fact, if λ is an eigenvalue with associated eigenfunction $\varphi(x)$, then

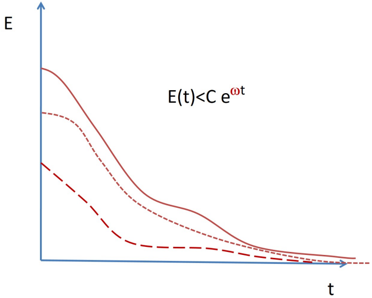
$$u(x,t) = e^{\lambda t}\varphi(x)$$

is a solution that decays as the real part of λ. Therefore,

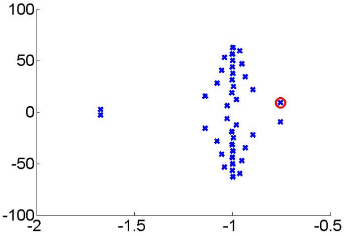
$$\omega(a) \geq \sup_{\lambda \in \sigma(A+B)} Re(\lambda) = \mu(a),$$

This last quantity $\mu(a)$ is known as the spectral abcissa.

$\omega(a) \geq \mu(a)$



decay rate: $\omega(a)$



spectral abcissa: $\mu(a)$

Motivation: the damped 1-d wave equation

When $a \in BV(0,1)$ both the spectral abcissa and the decay rate are the same (Cox-Zuazua, 93). The main idea is to prove that the eigenfunctions constitute a Riesz basis of the energy space.

Similar questions arise in other damped one-dimensional models (Schrödinger, beam, etc.) and in higher dimensions, where the spectral abcissa also plays a role (Lebeau, 96).

Main question: Find a numerical approximations of $\mu(a)$

Difficulties:

1. It requires an approximation of the whole spectra, and not only a finite number of eigenvalues.
2. The underlying operator is not skew-adjoint (or selfadjoint).

A general setting

Let H be a Hilbert space, $A : D(A) \to H$ unbounded skew adjoint operator and $B : H \to H$ bounded,

$$\begin{cases} y'(t) = Ay(t) + By(t), & t \geq 0, \\ y(0) = y_0 \in H \end{cases}$$

with

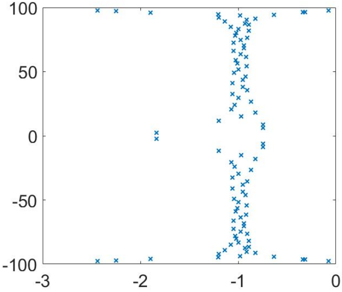
$$Re \ < By, y > \leq 0.$$

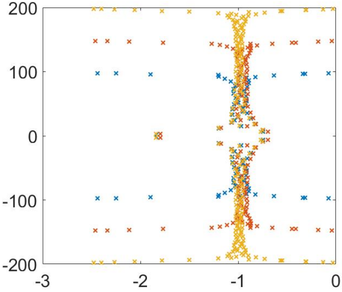
Under this setting we can consider wave, beam or Schrödinger type models in particular.

For some of these models, specially in 1-d, there exist results on the characterization of the decay rate with the spectral abcissa.

A first natural approach: the finite element method

Spectrum of the FEM approximation of the 1-d damped wave equation $a(x) = \chi_{(0.25,0.75)}(x)$. ($Re\ (\lambda) \to -1$ as $|\lambda| \to \infty$)

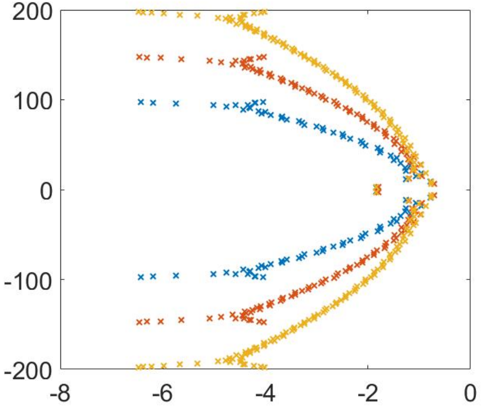




Convergence for a finite number of eigenvalues (Fix 73, Bamberger-Osborn 73, Vainnikko 70)
Approximation of the decay rate in 2d: (Asch-Lebeau 99)

Adding numerical viscosity

Numerical viscosity recover the exponential decay of discrete approximations (Tebou, Zuazua 07; Ramdani, Takahashi, Tucsnak 07; Marica, Zuazua 14)



The projection method

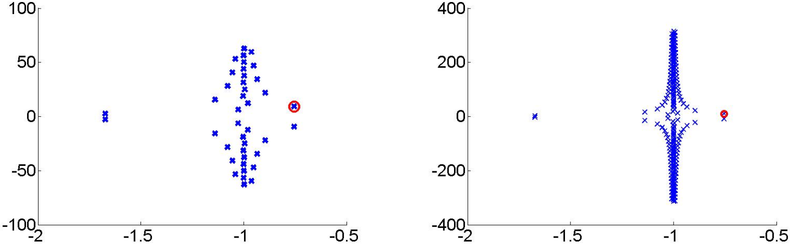
Idea: Project the eigenvalue problem in the finite dimensional vector space generated by the first N eigenfunctions of the unperturbed skew-adjoint operator.
Let $\left(V_{\pm k}\right)_{k\in\mathbb{N}^*}$ be the orthonormal basis of eigenfunctions of A.

$$H^N = span\{V_k\}_{k\in Z_N^*} \subset H$$

where $Z_N^* = \{k \in \mathbb{Z}^*,\ |k| \leq N\}$ and $P^N : H \to H^N$ the associated orthogonal projection.
Discrete eigenvalue problem: Find $\lambda^N \in \mathbb{C}$ such that there exists a solution $U^N \in H^N$, $U^N \neq 0$ of the system

$$P^N(A+B)U^N = \lambda U^N. \tag{2}$$

Numerical evidences of the approximation

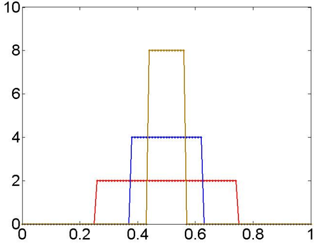


40 frequencies   200 frequencies

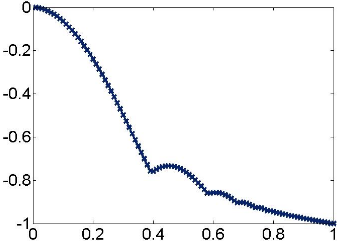
$$a(x) = \chi_{(0.25,0.75)}(x)$$

Some examples

Approximation of a Dirac at $x = 0.5$



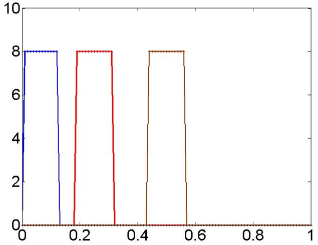
$a(x)$ (Dirac approximation)



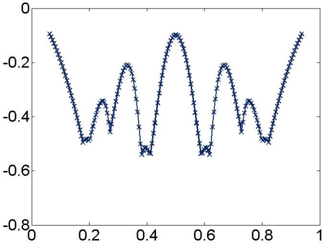
length of the support vs spectral abcissa (100 frequencies)

Some examples

Influence of the position of the damping



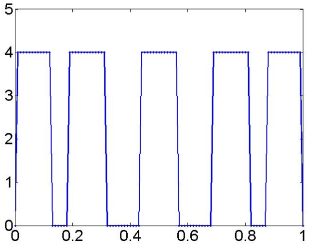
$a(x)$



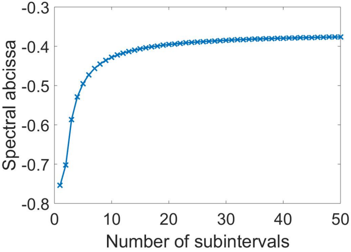
position of the support vs spectral abcissa (100 frequencies)

Some examples

An oscillating damping with average 2. $a(x/\varepsilon) \to 2$ weakly but the decay rate does not converge $\omega(a(x/\varepsilon)) > \omega(2) = -1$.



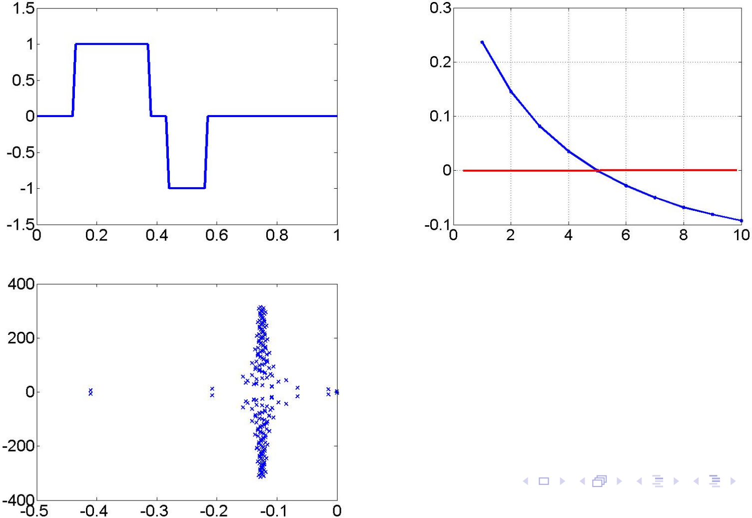
$a(x/\varepsilon)$



number of intervals vs
spectral abcissa (100 frequencies)

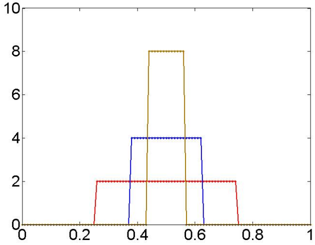
Some examples

Indefinite dampings. There is stabilization if the damping is not large.

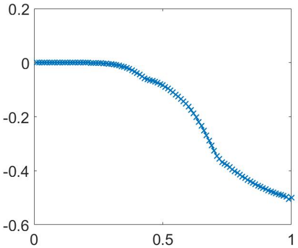


Some examples

Remark: 1-d Schrödinger and plate models have a similar behavior.
Consider the 2-d wave equation.

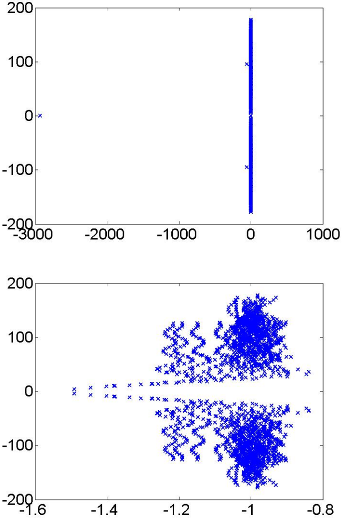


$a(x)$ (Dirac approximation)



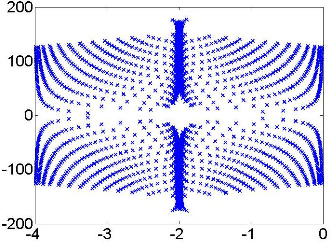
length of the support vs spectral abcissa (100 frequencies)

Some examples

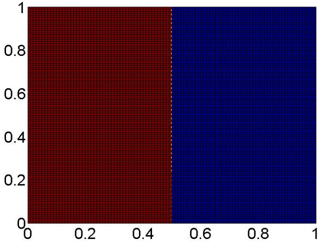


Some examples

An example in Asch-Lebeau (99), solved with finite elements,



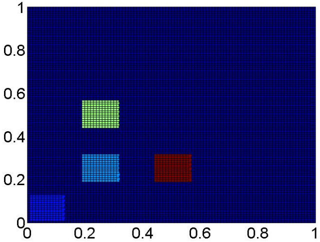
Spectrum



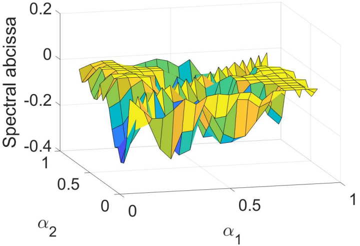
damping $\chi_{(x<1/2)}(x,y)$

Some examples

2-d wave equation. There is no uniform stabilization but the spectral abcissa can be negative!



$a(x)$ supported in a square



position of the support vs spectral abcissa (100 frequencies)

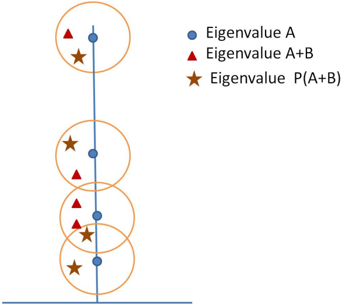
Main result

Notation:

A, (skewadjoint). Eigenvalues and eigenvectors: $(\lambda_k, V_k)_{k\geq 1}$

$A+B$, Eigenvalues and eigenvectors: $(\nu_k, U_k)_{k\geq 1}$

$P^N(A+B)$, Eigenvalues and eigenvectors: $(\nu_k^N, U_k^N)_{k\geq 1}$



Main result

Theorem Assume that the following hypotheses are satisfied:

H1 $A : D(A) \subset H \to H$ is skew-adjoint with simple eigenvalues $\{\lambda_j\}_{j\in\mathbb{Z}^*}$. (not for 2d problems)

H2 The eigenvalues of A satisfy an asymptotic gap

$$|\lambda_{j+1} - \lambda_j| \geq \delta > 0, \text{ for all } j \in \mathbb{Z}^*.$$

(Not for the 2-d wave equation in square $\lambda_{kl} = i\pi\sqrt{k^2 + l^2}$)

H3 $\|B\|_H \leq \lambda_1$

H4 The eigenvectors of $A + B$, $\{U_j\}_{j\in\mathbb{Z}^*}$ constitute a Riesz basis of H, i.e. there exists constants $m(B)$, $M(B) > 0$ such that

$$m(B)\sum_j |c_j|^2 \leq \left\| \sum_j c_j U_j \right\|^2 \leq M(B)\sum_j |c_j|^2, \quad \{c_j\} \in l^2(\mathbb{C}).$$

Then, if $\{\nu_j\}_{j\in\mathbb{Z}^*}$ are the eigenvalues of $A+B$ and $\{\nu_j^N\}_{j\in\mathbb{Z}_N^*}$ those of $P^N(A+B)$,

$$\min_j |\nu_p^N - \nu_j| \leq C(\|B\|)\sqrt{\frac{M(B)}{m(B)}}\varepsilon(B,r), \qquad \text{for all } |p| < N-r$$

$$\min_j |\nu_p - \nu_j^N| \leq C(\|B\|)\sqrt{\frac{M(B)}{m(B)}}\varepsilon(B,r), \qquad \text{for all } |p| < N-r$$

where

$$\varepsilon(B,r) = \max_i \sum_{j:|i-j|>r} |< BV_i, V_j >|^2$$

Remark Roughly speaking, this last quantity measures how diagonal is the matrix

$$(< BV_i, V_j >)_{i,j}.$$

Nondiagonal terms must be small to have good estimates.

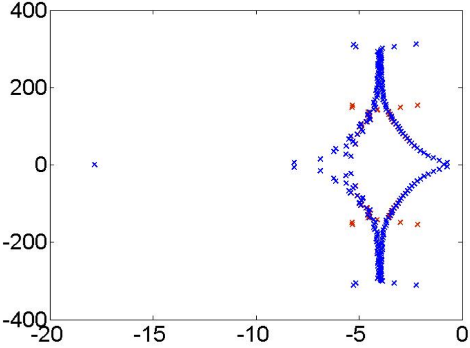
Remark For the 1-d damped wave, Schrödinger and beam models we have

$$< BV_i, V_j >= \int_0^1 a(x)\sin(i\pi x)\sin(j\pi x)dx,$$

and $\varepsilon(B, r)$ can be estimated by the L^1-norm of $a'(x)$.

Remark In practice we only have estimates for all the frequencies but the largest $2r$.

Last eigenvalues are not well approximated: $a(x) = 8\chi_{(0.1,0.5)}(x)$



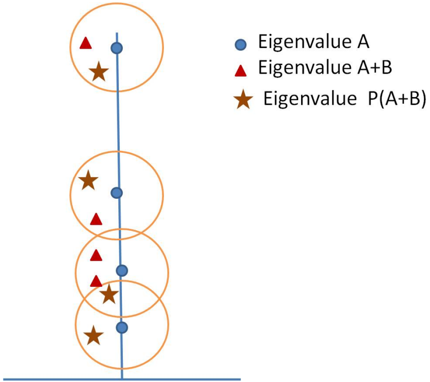
Approximation with 50 frequencies (red) and 100 frequencies (blue). Only the last few frequencies are not well approximated.

Sketch of the proof

The idea is to adapt the proof by Osborn (65) that consider the convergence of a finite number of eigenvalues.

Step 1. Basic estimates (Osborn 65). For each eigenvalue λ_j of A we consider

$$C_j = \{\lambda : |\lambda - \lambda_j| \leq \|B\|\}$$



Step 2. As the eigenfunctions of $A+B$ constitute a Riesz basis, we can write any eigenvector of $P^N(A+B)$ as a linear combination of them,

$$U_p^N = \sum_{j\in\mathbb{Z}^*} \alpha_{p,j} U_j, \quad \alpha_{p,j} \in \mathbb{C}.$$

where U_p^N is the eigenvector of $P^N(A+B)$ associated to ν_p^N. Then, after some algebra

$$\sum_{j\in\mathbb{Z}^*} \alpha_{p,j} U_j(\nu_p^N - \nu_j) = [P^N(A+B) - (A+B)]U_p^N = (I - P^N)BU_p^N$$

Taking norms and using the Riesz basis estimates,

$$\min_j |\nu_p^N - \nu_j| \leq \sqrt{\frac{M(B)}{m(B)}} \|(I - P^N)BU_p^N\|_H, \quad \text{for all } |p| \leq N$$

Step 3. We have to estimate,

$$\|(I - P^N)BU_p^N\|_H,$$

Hipothesis (H2) on the asymptotic spectral gap implies

$$U_p^N \sim V_p, \qquad \text{for } p \text{ large but still } |p| \leq N$$

since Fourier coefficients decay very fast.
This implies that basically

$$\|(I - P^N)BU_p^N\|_H \sim \|(I - P^N)BV_p\|_H, \quad |p| \leq N$$

Note that if B is diagonal this is zero. Therefore, it is natural to bound this quantity by a deviation

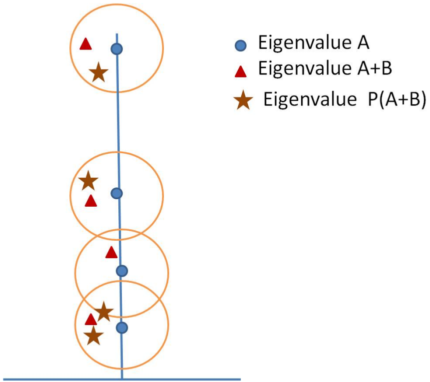
$$\varepsilon(B, r) = \max_i \sum_{j:|i-j|>r} | < BV_i, V_j > |^2$$

but only when $p < N - r$.

Step 4 So far we have estimated

$$\min_j |\nu_p^N - \nu_j|, \quad |p| \leq N - r$$

This means that there is an eigenvalue of $A + B$ close to one of $P^N(A + B)$. But, is there an eigenvalue of $A + B$ that is not close to an eigenvalue of $P^N(A + B)$? You can argue on a finite number of eigenvalues



Conclusions

1. The projection method provides a uniform convergence of the spectra, up to a finite number of high frequencies, for a large class of bounded perturbation of unbounded skew-adjoint operators.
2. The Theorem only covers a particular class of systems but there are numerical evidences that indicate its validity in more general situations.
3. The main drawback is that it requires the knowledge of the eigenfunctions of the unperturbed operator. For higher dimensions in general domains this is a difficult problem.