On the collision invariants of Boltzmann-like kinetic equations

Maxime Breden

Technische Universität München

Joint work with Laurent Desvillettes

Qualitative behaviour of kinetic equations and related problems: numerical and theoretical aspects, HIM

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Outline

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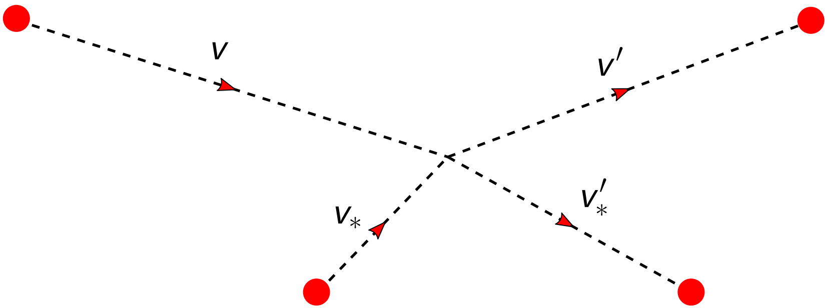
Binary elastic collisions



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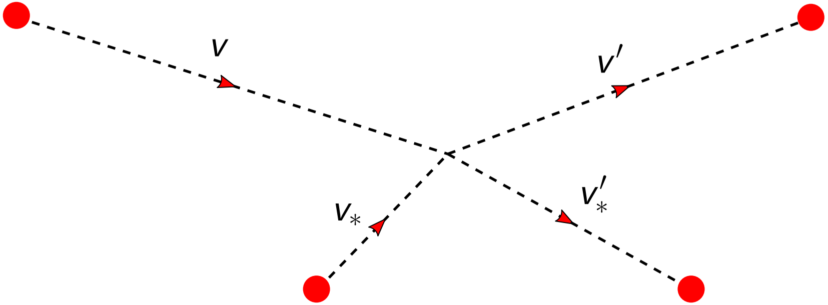

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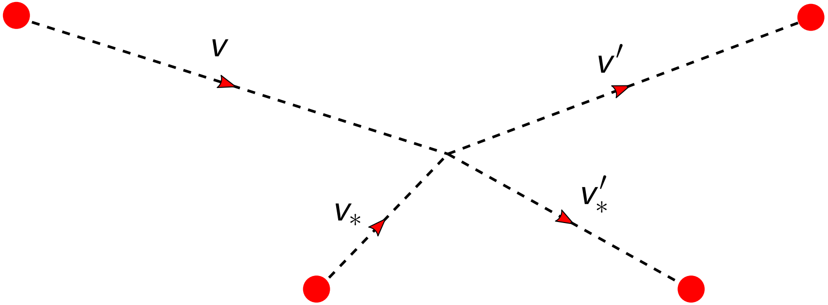

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- We say that a quadruple of velocities $(v, v_*, v', v'_*) \in (\mathbb{R}^d)^4$ is *admissible* if these two equalities are satisfied.

The Boltzmann equation

- Assuming we have a large number of (identical) particles, the whole system can be described by the density $f = f(t, x, v)$ of particles, depending on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^d$, and velocity $v \in \mathbb{R}^d$.

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$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma \end{cases} \qquad \sigma \in \mathbb{S}^{d-1}.$$

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- What are the functions $\varphi = \varphi(v)$ such that

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$$\int Q(f)(v)\varphi(v)dv = -\frac{1}{4}\int_{\substack{\text{all admissible}\\ \text{collision velocities}}} B\left(f(v')f(v'_*) - f(v)f(v_*)\right) \times \left(\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*)\right) dv dv_* dv' dv'_*.$$

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- Therefore, the collision invariants also characterize the kernel of the linearized operator.

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- Assuming φ is twice differentiable [Boltzmann 1875].
- Assuming φ is continuous [Gronwall 1915; Carleman 1957].
- Assuming φ is L^1_{loc} [Arkeryd and Cercignani 1990].

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- In all those cases, the collision invariants play an important role, and are related to the thermodynamic equilibria via

$$\varphi(p) = \ln\left(\frac{f(p)}{1+\varepsilon f(p)}\right).$$

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$$Q_{\text{waves}}(f)(k) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} B\left(f'f'_*(f+f_*) - ff_*(f'+f'_*)\right)$$
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- For gravity waves, the dispersion law is of the form

$$\omega(k) = \sqrt{|k|}.$$

A brief (and non-exhaustive) literature review

- The relativistic Boltzmann equation has been studied quite extensively [Cercignani and Kremer 2002; Escobedo, Mischler and Valle 2003].

- The kinetic equations appearing in the theory of weak turbulence have been mostly studied in the physics literature [Zakharov, L'vov, Falkovich 1992]. More recently, some attempt have been made to rigorously derive these kinetic equations [Lukkarinen and Spohn 2011], and the Cauchy problem has also been studied [Germain, Ionescu and Tran 2017].

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► If $\varphi \in \mathcal{C}^1(\mathbb{R}^d_*, \mathbb{R})$ satisfies

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then,

$$\varphi(v) = a + b \cdot v + c\, \omega(v),$$

for some constants $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

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L^1_{loc} case [B. and Desvillettes 2018]

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- Extra hypothesis: Assume the boundary of $\{(v,v_*) \mid \nabla\omega(v) \neq \nabla\omega(v_*)\}$ is of measure zero.

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- If $\nabla\omega(v) \neq \nabla\omega(v_*)$, then locally, the set of admissible collision velocities can be parametrized by:

$$z = \gamma(v, v_*, \sigma), \quad \sigma \in \mathbb{R}^{d-1},$$

with $\gamma(v, v_*, 0) = 0$ and $\mathrm{rank}\left(D_\sigma\gamma(v, v_*, 0)\right) = d - 1$.

Parameterization of the admissible collision velocities

- The admissible collision velocities can be reduced to the (v, v_*, z) s.t.

$$\omega(v) + \omega(v_*) = \omega(v - z) + \omega(v_* + z),$$

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▶ In the L^1_{loc} case, we assume that, for almost all $v, v_* \in \mathbb{R}^d_*$ such that $\nabla\omega(v) \neq \nabla\omega(v_*)$, and almost all $\sigma \in \mathbb{R}^{d-1}$ in a neighbourhood of 0,

$$\varphi(v) + \varphi(v_*) = \varphi(v - \gamma(v, v_*, \sigma)) + \varphi(v_* + \gamma(v, v_*, \sigma)).$$

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In coordinates, we have just shown that

$$(\partial_i\varphi(v) - \partial_i\varphi(v_*))\,(\partial_j\omega(v) - \partial_j\omega(v_*)) = \\ (\partial_j\varphi(v) - \partial_j\varphi(v_*))\,(\partial_i\omega(v) - \partial_i\omega(v_*))\,. \quad (\star)$$

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- The main idea, based on [Desvillettes 2015], is then to test $(\star)$ against well chosen functions:

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$$\int (\star)\; dv_*, \qquad \int (\star)\; \partial_i\omega(v_*) dv_*, \qquad \int (\star)\; \partial_j\omega(v_*) dv_*.$$

- The resulting equalities can be rewritten as a linear system $Ax = y$, where

$$A = \begin{pmatrix} \int dv_* & \int \partial_i\omega(v_*) dv_* & \int \partial_j\omega(v_*) dv_* \\ \int \partial_i\omega(v_*) dv_* & \int \left(\partial_i\omega(v_*)\right)^2 dv_* & \int \partial_i\omega(v_*)\partial_j\omega(v_*) dv_* \\ \int \partial_j\omega(v_*) dv_* & \int \partial_i\omega(v_*)\partial_j\omega(v_*) dv_* & \int \left(\partial_j\omega(v_*)\right)^2 dv_* \end{pmatrix},$$

$$x = \begin{pmatrix} \partial_j\varphi(v)\partial_i\omega(v) - \partial_i\varphi(v)\partial_j\omega(v) \\ -\partial_j\varphi(v) \\ \partial_i\varphi(v) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 + y_2\partial_i\omega(v) + y_3\partial_j\omega(v) \\ y_4 + y_5\partial_i\omega(v) + y_6\partial_j\omega(v) \\ y_7 + y_8\partial_i\omega(v) + y_9\partial_j\omega(v) \end{pmatrix}.$$

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- Since 1, $\partial_i \omega$ and $\partial_j \omega$ are linearly independent, the Gram matrix A is invertible, and $x = A^{-1} y$, which yields

$$\begin{cases} \partial_i \varphi(v) = b_i + c_i \partial_i \omega(v) + d_i \partial_j \omega(v) \\ \partial_j \varphi(v) = b_j + c_j \partial_j \omega(v) + d_j \partial_i \omega(v) \end{cases}$$

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- Plugging this back in $(\star)$, and using once more the linear independence of 1, $\partial_i\omega$ and $\partial_j\omega$, we get

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- Therefore $\nabla\varphi(v) = b + c\nabla\omega(v)$ and thus

$$\varphi(v) = a + b\cdot v + c\omega(v). \quad \square$$

Some counterexamples

Some counterexamples

- In dimension 1 (not covered by our results), there are many functions ω for which the set of admissible collision velocities is trivial. For instance, if ω is strictly concave or convex, then the only (v, v_*, v', v'_*) satisfying

$$v + v_* = v' + v'_* \qquad \text{and} \qquad \omega(v) + \omega(v_*) = \omega(v') + \omega(v'_*)$$

are of the form (v, v_*, v, v_*) and (v, v_*, v_*, v), and thus

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- In dimension $d = 2$, if 1, $\partial_i\omega$ and $\partial_j\omega$ are NOT linearly independent, then up to a change of coordinate ω is of the form

$$\omega(v) = g(v_1) + \beta v_2.$$

Therefore, as soon as the set of admissible collision velocities is trivial for g, any function φ of the form

$$\varphi(v) = h(v_1) + bv_2$$

is a collision invariant.

Back to the equilibria

When 1, $\partial_i\omega$ and $\partial_j\omega$ are linearly independent, the collision invariant must be of the form

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- For the Boltzmann equation for waves, the relation is given by

$$f_{\mathrm{eq}}(v) = \frac{1}{\varphi(v)}.$$

Outline

3-waves interactions

- In the example discussed up to now for kinetic equations for waves, we only studied the so-called *4-waves interactions*, characterized by

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- For other dispersion relations, for example $\omega(v) = |v|^{3/2}$ corresponding to capillary waves, these 3-waves interactions do occur, and they lead to a slightly different kinetic equation.

The 3-waves collision kernel

- The collision operator for these 3-waves interaction is given by

$$\tilde{Q}_3(f)(v) = \int \left(R(v, v_*, v') - R(v_*, v, v') - R(v', v, v_*)\right) dv_* dv',$$

where

$$R(v, v_*, v') = B(v, v_*, v') \left(f(v_*)f(v') - f(v)(f(v_*) + f(v'))\right) \\ \times \delta_{\{v=v_*+v'\}}\delta_{\{\omega(v)=\omega(v_*)+\omega(v')\}}.$$

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$$\varphi(v) + \varphi(v_*) = \varphi(v'),$$

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Partial results [B. and Desvillettes 2018]

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▶ If $\varphi \in \mathcal{C}^1(\mathbb{R}^d_*, \mathbb{R})$ satisfies

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for all (v, v_*, v') such that

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$$\varphi(v) = b \cdot v + c\,\omega(v),$$

for some constants $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$.

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- Conclude by showing that γ is in fact linear.

Comments, counterexamples and open questions

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- Very interestingly, there exists another physically relevant dispersion law

$$\omega(v) = \frac{v_1}{1 + |v|^2},$$

corresponding to Rossby waves (also called planetary waves), which admits an extra collision invariant [Balk 1991]:

$$\varphi(v) = \arctan\left(\frac{v_1\sqrt{3} + v_2}{|v|^2}\right) + \arctan\left(\frac{v_1\sqrt{3} - v_2}{|v|^2}\right).$$

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- This example violates several assumptions of our theorem at once. Therefore it is not clear which are the properties of ω (or φ) allowing for this extra invariant to exist!

THANK YOU FOR YOUR ATTENTION!