Hypocoercivity and diffusion limit of a finite volume scheme for linear kinetic equations

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Linear kinetic equation

For $t \geq 0$, $x \in \mathbb{T}$ and $v \in \mathbb{R}$,

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} \;=\; \mathcal{Q}(f), \\ \\ f(0,x,v) = f_0(x,v) \geq 0. \end{cases}$$

Mass preserving collision operators :

$$\mathcal{Q}_{FP}(f)(v) = \partial_v \left(\partial_v f + v f\right),$$

or

$$\mathcal{Q}_{BGK}(f)(v) = \rho\mathcal{M}(v) - m_0 f(v),$$

where $\mathcal{M}$ is the Maxwellian, and ρ the local mass of f, namely

$$\rho = \int_{\mathbb{R}} f(v)\,\mathrm{d}v.$$

Hypocoercivity

Exponential return to equilibrium : there exists an appropriate functional space $\mathcal{X}$ and constants $\kappa > 0$, $C \geq 1$ such that

$$\|f(t) - m_f \mathcal{M}\|_{\mathcal{X}} \leq C \, \|f(0) - m_f \mathcal{M}\|_{\mathcal{X}} \, e^{-\kappa t},$$

where $m_f = \int \int f \, \mathrm{d}x \, \mathrm{d}v$.

A very brief history :

- **Hérau** (2006)
- **Villani** (2009)
- **Dolbeault, Mouhot, Schmeiser** (2015)

Diffusive limit

Parabolic scaling : $t \to t/\varepsilon^2$, $x \to x/\varepsilon$

$$\Rightarrow \varepsilon \frac{\partial f^\varepsilon}{\partial t} + v \frac{\partial f^\varepsilon}{\partial x} \; = \; \frac{1}{\varepsilon} \, \mathcal{Q}(f^\varepsilon).$$

Moments equations :

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x J^\varepsilon = 0, \\ \varepsilon^2 \partial_t J^\varepsilon + m_2 \partial_x \rho^\varepsilon + \partial_x S^\varepsilon = -J^\varepsilon, \end{cases}$$

where

$$\rho^\varepsilon := \int_{\mathbb{R}} f^\varepsilon \, \mathrm{d}v, \quad J^\varepsilon := \frac{1}{\varepsilon} \int_{\mathbb{R}} f^\varepsilon \, v \, \mathrm{d}v, \quad S^\varepsilon := \int_{\mathbb{R}} (v^2 - m_2) f^\varepsilon \, \mathrm{d}v.$$

Diffusive limit $\varepsilon \to 0$: $f^\varepsilon \to f := \rho \mathcal{M}$, $S^\varepsilon \to 0$ and $J^\varepsilon \to -m_2 \partial_x \rho$, with ρ solution of

$$\frac{\partial \rho}{\partial t} - \partial_x (m_2 \partial_x \rho) = 0.$$

- **Degond, Goudon, Poupaud** (2000) : linear kinetic equations.
- **Mellet, Mischler, Mouhot** (2011) : fractional diffusion limits.

Asymptotic Preserving schemes

AP schemes for kinetic equations :

- **Klar** (1998) : linear BGK equation for semiconductors,
- **Jin, Pareschi, Toscani** (2000) : relaxation models,
- **Jin** (2010), **Dimarco, Pareschi** (2014) : review papers,
- ...

Proofs of AP properties :

- **Lemou, Mieussens** (2008) : micro-macro approach,
- **Liu, Mieussens** (2010) : detailed stability analysis,
- **Crouseilles, Hivert, Lemou** (2016) : anomalous diffusion, time discretization,
- **Dujardin, Hérau, Lafitte** (2018) : hypocoercivity for full discrete FP equation,
- ...

Aims

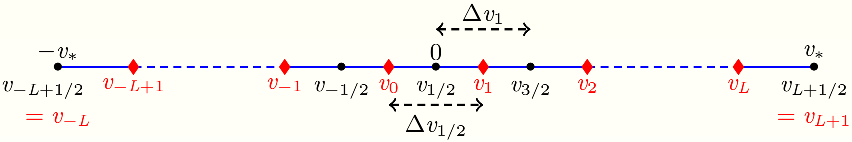
Develop numerical schemes for linear kinetic equations which

- mimic continuous behaviour (energy estimates, entropy–entropy dissipation inequalities, hypocoercivity),
- are stable for all ε and have the correct diffusion limit.

Mesh

- Discretization of the bounded velocity domain $[-v_*, v_*]$ into $2L$ cells :

$$\mathcal{V}_j = (v_{j-\frac{1}{2}}, v_{j+\frac{1}{2}}), \quad j \in \mathcal{J} = \{-L+1, \dots, L\}.$$



Dual mesh : $\mathcal{V}^*_{j+\frac{1}{2}} = (v_j, v_{j+1}), \quad j \in \mathcal{J}^* = \{-L, \dots, L\}.$

- Discretization of the spatial domain $\mathbb{T}$ into N subintervals :

$$\mathcal{X}_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad i \in \mathcal{I} = \mathbb{Z}/N\mathbb{Z}.$$

- Time step $\Delta t > 0$, $t^n = n\Delta t$ for all $n \geq 0$.

Discrete Maxwellians

BGK case. We define cell values $\mathcal{M}^\Delta = (\mathcal{M}_j)_{j\in\mathcal{J}}$ such that

$$\begin{cases} \mathcal{M}_j > 0\,, \quad \mathcal{M}_j \;=\; \mathcal{M}_{-j+1}\,, \quad \forall j = 1,\ldots,L\,, \\ \displaystyle\sum_{j\in\mathcal{J}} \mathcal{M}_j\,\Delta v_j \;=\; 1\,, \\ 0 \;<\; \underline{m_2} \;\leq\; m_2^{\Delta v} \;\leq\; \overline{m_2}\,, \quad m_4^{\Delta v} \;\leq\; \overline{m_4}\,. \end{cases}$$

Fokker Planck case. We define interface values $(\mathcal{M}^*_{j+1/2})_{j\in\mathcal{J}^*} \in \mathbb{R}^{\mathcal{J}^*}$ such that

$$\begin{cases} \mathcal{M}^*_{j+1/2} \;=\; \mathcal{M}^*_{-j+1/2}\,, \quad \forall j \in \mathcal{J}^*\,, \\ \mathcal{M}^*_{L+1/2} \;=\; \mathcal{M}^*_{-L+1/2} \;=\; 0\,, \\ \mathcal{M}_j \;:=\; \dfrac{\mathcal{M}^*_{j-1/2} - \mathcal{M}^*_{j+1/2}}{v_j\,\Delta v_j} \;>\; 0\,, \quad \forall j \in \mathcal{J}\,; \\ \displaystyle\sum_{j\in\mathcal{J}} \mathcal{M}_j\,\Delta v_j \;=\; 1\,, \\ 0 \;<\; \underline{m_2} \;\leq\; m_2^{\Delta v} \;\leq\; \overline{m_2}\,, \quad m_4^{\Delta v} \;\leq\; \overline{m_4}\,. \end{cases}$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \left(\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{n+1} \right),$$

where the free transport fluxes $\mathcal{F}_{i+\frac{1}{2},j}$ and the Fokker-Planck fluxes $\mathcal{G}_{i,j+\frac{1}{2}}$ are :

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} \approx \int_{\mathcal{V}_j} v f(t^{n+1}, x_{i+\frac{1}{2}}, v) \, \mathrm{d}v \,,$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{\mathcal{X}_i} \left(\partial_v f + v_{j+\frac{1}{2}} f \right) (t^{n+1}, x, v_{j+\frac{1}{2}}) \, \mathrm{d}x \,.$$

Fully discrete Fokker-Planck equation

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where the free transport fluxes $\mathcal{F}_{i+\frac{1}{2},j}$ and the Fokker-Planck fluxes $\mathcal{G}_{i,j+\frac{1}{2}}$ are :

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = \ v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{\mathcal{X}_i} \left(\partial_v f + v_{j+\frac{1}{2}} f \right) (t^{n+1}, x, v_{j+\frac{1}{2}}) \, \mathrm{d}x \, .$$

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$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = \ v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{\mathcal{X}_i} \mathcal{M}\, \partial_v \left(\frac{f}{\mathcal{M}} \right) (t^{n+1}, x, v_{j+\frac{1}{2}})\, \mathrm{d}x\,.$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \left(\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{n+1} \right),$$

where the free transport fluxes $\mathcal{F}_{i+\frac{1}{2},j}$ and the Fokker-Planck fluxes $\mathcal{G}_{i,j+\frac{1}{2}}$ are :

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} = \mathcal{M}_{j+\frac{1}{2}}^* \left(\frac{f_{i,j+1}^{n+1}}{\mathcal{M}_{j+1}} - \frac{f_{ij}^{n+1}}{\mathcal{M}_j} \right) \frac{1}{\Delta v_{j+\frac{1}{2}}} \Delta x_i, \quad \forall j \in \mathcal{J}^* \setminus \{-L, L\}, \quad \forall i \in \mathcal{I}.$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \left(\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{n+1} \right),$$

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$$\mathcal{G}_{i,-L+\frac{1}{2}}^{n+1} = \mathcal{G}_{i,L+\frac{1}{2}}^{n+1} = 0, \quad \forall i \in \mathcal{I}.$$

Fully discrete BGK equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^{n}) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \Delta x_i \Delta v_j \left(\rho_i^{n+1} \mathcal{M}_j - f_{ij}^{n+1} \right),$$

where

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

and

$$\rho_i^{n+1} := \sum_{j \in \mathcal{J}} \Delta v_j f_{ij}^{n+1} \quad \forall i \in \mathcal{I}.$$

Discrete “entropy” estimate

Proposition

Let $(f_{ij}^n)_{i\in\mathcal{I},j\in\mathcal{J},n\in\mathbb{N}}$ solve the scheme either in Fokker Planck or BGK case. Then for all $\varepsilon > 0$, and for every $n \geq 0$,

$$\frac{\|f^{n+1}\|_{2,\gamma}^2 - \|f^n\|_{2,\gamma}^2}{2\Delta t} + \frac{1}{\varepsilon^2}\|f^{n+1} - \rho^{n+1}\mathcal{M}^\Delta\|_{2,\gamma}^2 \leq 0\,.$$

In particular one has

$$\max\left(\sup_{n\geq 0}\|f^n\|_{2,\gamma}^2,\ \frac{2}{\varepsilon^2}\sum_{n=1}^{\infty}\Delta t\,\|f^n - \rho^n\mathcal{M}^\Delta\|_{2,\gamma}^2\right) \leq \|f^0\|_{2,\gamma}^2\,.$$

Consequence : existence and uniqueness of the discrete solution.

Proof

Multiplying equation by $f/\mathcal{M}$, and integrating in x and v yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} + \frac{1}{2\varepsilon}\iint \partial_x\left(v\,\frac{f^2}{\mathcal{M}}\right)\mathrm{d}x\mathrm{d}v = \frac{1}{\varepsilon^2}\iint \mathcal{Q}(f)\frac{f}{\mathcal{M}}\mathrm{d}v\mathrm{d}v.$$

Proof

Multiplying equation by $f/\mathcal{M}$, and integrating in x and v yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} + 0 = \frac{1}{\varepsilon^2}\iint \mathcal{Q}(f)\frac{f}{\mathcal{M}}\mathrm{d}v\mathrm{d}v.$$

BGK case : Using $\int \mathcal{Q}_{BGK}(f)\mathrm{d}v = 0$,

$$\Rightarrow \iint \mathcal{Q}_{BGK}(f)\frac{f}{\mathcal{M}} = \iint \mathcal{Q}_{BGK}(f)\left(\frac{f}{\mathcal{M}} - \rho\right) = -\|f - \rho\mathcal{M}\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)}.$$

Fokker Planck case : Using **Gaussian Poincaré inequality**,

$$\iint \mathcal{Q}_{FP}(f)\frac{f}{\mathcal{M}} = -\left\|\partial_v\left(\frac{f}{\mathcal{M}}\right)\right\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mathcal{M})} \leq -\|f - \rho\mathcal{M}\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\gamma)}.$$

Discrete moments estimates

$$J_i^n \ := \ \frac{1}{\varepsilon} \sum_{j\in\mathcal{J}} \Delta v_j \, v_j \, f_{ij}^n \,, \qquad S_i^n \ := \ \sum_{j\in\mathcal{J}} \Delta v_j \, (v_j^2 - m_2^{\Delta v}) \, f_{ij}^n \,, \quad \forall i \in \mathcal{I}, \quad \forall n \geq 0.$$

Lemma

The moments $(\rho_i^n)_{i\in\mathcal{I}}$, $(J_i^n)_{i\in\mathcal{I}}$ and $(S_i^n)_{i\in\mathcal{I}}$ satisfy :

$$\|\rho^n\|_2 \ \leq \ \|f^n\|_{2,\gamma} \,,$$

$$\varepsilon \, \|J^n\|_2 \ \leq \ (m_2^{\Delta v})^{1/2} \, \|f^n\|_{2,\gamma} \,,$$

$$\varepsilon \, \|J^n\|_2 \ \leq \ (m_2^{\Delta v})^{1/2} \, \|f^n - \rho^n \, \mathcal{M}^\Delta\|_{2,\gamma} \,,$$

$$\|S^n\|_2 \ \leq \ (m_4^{\Delta v} - (m_2^{\Delta v})^2)^{1/2} \, \|f^n - \rho^n \, \mathcal{M}^\Delta\|_{2,\gamma} \,.$$

In particular, $\exists C > 0$ only depending on the bounds on $m_2^{\Delta v}$, $m_4^{\Delta v}$ such that

$$\sup_{n\in\mathbb{N}} \|\rho^n\|_2^2, \ \varepsilon^2 \sup_{n\in\mathbb{N}} \|J^n\|_2^2, \ \sum_{n=0}^{\infty} \Delta t \, \|J^n\|_2^2, \ \frac{1}{\varepsilon^2} \sum_{n=0}^{\infty} \Delta t \, \|S^n\|_2^2 \leq \ C \, \|f_0\|_{2,\gamma}^2 \,.$$

Discrete moments equations

Lemma

Let us consider a solution to the scheme. Then the discrete moments satisfy the following equations. For all $i \in \mathcal{I}$, $n \geq 0$,

$$\Delta x_i \, (\rho_i^{n+1} - \rho_i^n) \, + \, \Delta t \, (J_{i+\frac{1}{2}}^{n+1} - J_{i-\frac{1}{2}}^{n+1}) = 0,$$

$$\varepsilon^2 \, \Delta x_i \, (J_i^{n+1} - J_i^n) \, + \, \Delta t \, (S_{i+\frac{1}{2}}^{n+1} - S_{i-\frac{1}{2}}^{n+1}) \, + \, \Delta t \, m_2^{\Delta v} \, (\rho_{i+\frac{1}{2}}^{n+1} - \rho_{i-\frac{1}{2}}^{n+1}) = -\Delta t \, \Delta x_i \, J_i^{n+1},$$

where

$$X_{i+\frac{1}{2}} = \frac{X_i + X_{i+1}}{2}, \quad X = \rho, \, J, \, S.$$

Asymptotic-preserving property

Theorem

Let $f_\varepsilon^n = (f_{ij}^n)_{i\in\mathcal{I},j\in\mathcal{J}}$ for $n \in \mathbb{N}$ be a solution of the scheme. Then there is $\rho^n = (\rho_i^n)_{i\in\mathcal{I}}$ for all $n \geq 0$ such that when $\varepsilon \to 0$ one has

$$f_\varepsilon^n \longrightarrow \rho^n \mathcal{M}^\Delta \quad \text{in} \quad \mathbb{R}^{\mathcal{I}\times\mathcal{J}}, \quad \text{for all} \quad n \geq 1,$$

and the limit ρ satisfies the following scheme for the heat equation

$$\Delta x_i \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{m_2^{\Delta v}}{2} \left((D_x\rho)_{i+1}^{n+1} - (D_x\rho)_{i-1}^{n+1} \right), \quad \forall i \in \mathcal{I},$$

with initial data $\rho_i^0 = \sum_{j\in\mathcal{J}} \Delta v_j f_{ij}^0$.

Proof

- A priori estimates ⇒ up to a subsequence $\varepsilon_k \to 0$,

$$\rho^n_{\varepsilon_k} \to \rho^n \quad \forall n \geq 0,$$
$$f^n_{\varepsilon_k} \to (\rho^n_i \mathcal{M}_j)_{i\in\mathcal{I},\, j\in\mathcal{J}} \quad \forall n \geq 1.$$

- Combining moments equations implies

$$\Delta x_i \,(\rho_i^{n+1} - \rho_i^n) \;=\; m_2^{\Delta v}\, \Delta t\, \left((D_x\rho^{n+1})_{i+1/2} - (D_x\rho^{n+1})_{i-1/2}\right)$$
$$+\,\Delta t\, \left((D_x S^{n+1})_{i+1/2} - (D_x S^{n+1})_{i-1/2}\right) + \varepsilon^2\,(J^{n+1}_{i+\frac12} - J^{n+1}_{i-\frac12}) - \varepsilon^2\,(J^{n}_{i+\frac12} - J^{n}_{i-\frac12}).$$

Proof

- A priori estimates $\Rightarrow$ up to a subsequence $\varepsilon_k \to 0$,

$$\rho^n_{\varepsilon_k} \to \rho^n \quad \forall n \geq 0,$$
$$f^n_{\varepsilon_k} \to (\rho^n_i \mathcal{M}_j)_{i\in\mathcal{I},\, j\in\mathcal{J}} \quad \forall n \geq 1.$$

- Combining moments equations implies

$$\Delta x_i \left(\rho_i^{n+1} - \rho_i^n\right) \;=\; m_2^{\Delta v}\, \Delta t\, \left((D_x\rho^{n+1})_{i+1/2} - (D_x\rho^{n+1})_{i-1/2}\right).$$

Proof

- A priori estimates $\Rightarrow$ up to a subsequence $\varepsilon_k \to 0$,

$$\rho^n_{\varepsilon_k} \to \rho^n \quad \forall n \geq 0,$$
$$f^n_{\varepsilon_k} \to (\rho^n_i \mathcal{M}_j)_{i\in\mathcal{I},\, j\in\mathcal{J}} \quad \forall n \geq 1.$$

- Combining moments equations implies

$$\Delta x_i \left(\rho_i^{n+1} - \rho_i^n\right) \;=\; m_2^{\Delta v}\, \Delta t\, \left((D_x\rho^{n+1})_{i+1/2} - (D_x\rho^{n+1})_{i-1/2}\right).$$

- The limit scheme has a unique solution $\Rightarrow$ the whole sequence converges.

Main result

Theorem

Assume that the number of points N in the space discretization is odd.

Then there are constants $C \geq 1$ and $\kappa > 0$ such that for all $\varepsilon \in (0,1)$, all $\Delta t \leq \Delta t_{\max}$ and all initial data $(f_{ij}^0)_{i\in\mathcal{I},j\in\mathcal{J}}$, the solution $(f_{ij}^n)_{i\in\mathcal{I},j\in\mathcal{J},n\in\mathbb{N}}$ of the scheme satisfies

$$\|f^n - m_f\mathcal{M}^\Delta\|_{2,\gamma} \ \leq \ C\,\|f^0 - m_f\mathcal{M}^\Delta\|_{2,\gamma}\, e^{-\kappa t^n}\,,$$

where $m_f = \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} \Delta x_i \Delta v_j f_{ij}^0$.

Moreover, the constants C and κ do not depend on the size of the discretization, and $\Delta t_{\max} > 0$ can be chosen arbitrarily.

Modified entropy

Up to changing f by $f - m_f \mathcal{M}$, we can assume that $m_f = 0$.

Modified entropy (**Dolbeault, Mouhot, Schmeiser, 2015**)

$$H(f^\varepsilon(t)) = \frac{1}{2} \|f^\varepsilon(t)\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} + \eta\,\varepsilon^2 \left\langle J^\varepsilon, \partial_x \phi^\varepsilon \right\rangle_{L^2_x} \,,$$

where $\phi^\varepsilon(t,x)$ is the solution of the Poisson equation

$$-\partial^2_{xx}\phi^\varepsilon \;=\; \rho^\varepsilon\,, \qquad \int_{\mathbb{T}} \phi^\varepsilon \mathrm{d}x \,=\, 0\,,$$

and $\eta > 0$ is a small parameter to be chosen.

Outline of proof

1. Entropy–dissipation estimate :

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f^\varepsilon) + K_\eta \left(\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} + \|\rho^\varepsilon \mathcal{M}\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)}\right) \leq 0\,,$$

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}H(f^\varepsilon) + \frac{K_\eta}{2}\|f^\varepsilon\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} \leq 0.$$

2. Control of the entropy :

$$\frac{c_\eta}{2}\,\|f^\varepsilon\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)} \leq H(f^\varepsilon) \leq \frac{C_\eta}{2}\,\|f^\varepsilon\|^2_{L^2(\mathrm{d}x\mathrm{d}\gamma)}\,.$$

$\Longrightarrow$ Exponential decay of $H(t)$, and then of $\|f^\varepsilon(t)\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}$.

Adaptation to the discrete framework

Modified discrete entropy

$$\mathbb{H}(f^n) := \frac{1}{2}\|f^n\|_{2,\gamma}^2 + \eta\varepsilon^2 \sum_{i\in\mathcal{I}} \Delta x_i J_i^n (D_x\phi)_i^n + \frac{\eta\varepsilon^2}{2} \sum_{i\in\mathcal{I}} \Delta x_i \frac{\left((D_x\phi)_i^n - (D_x\phi)_i^{n-1}\right)^2}{\Delta t},$$

where $(\phi_i^n)_{i\in\mathcal{I}}$ is the solution of the discrete Poisson equation

$$-\frac{(D_x\phi)_{i+1}^n - (D_x\phi)_{i-1}^n}{2} = \Delta x_i \rho_i^n, \quad \forall i \in \mathcal{I},$$

$$\sum_{i\in\mathcal{I}} \Delta x_i \phi_i^n = 0,$$

with

$$(D_x\phi)_i = \frac{\phi_{i+1} - \phi_{i-1}}{\Delta x_i}.$$

N odd $\Rightarrow$ existence and uniqueness of $(\phi_i^n)_{i\in\mathcal{I}}$.

$\Rightarrow$ discrete Poincaré inequality on $\mathbb{T}$.

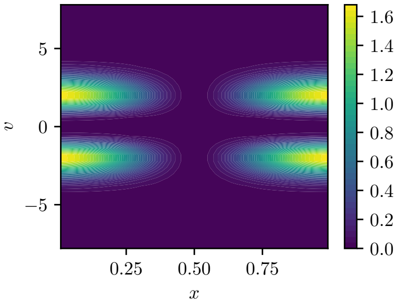
Data

Discretization parameters.

- $v_* = 8$, $L = 20$ ($\Rightarrow$ 40 cells in velocity),
- $N = 51$ cells in the torus,
- $\Delta t = 0.1$.

Initial data.

$$f_0(x,v) := \frac{1}{\sqrt{2\pi}}\, v^4\, e^{-v^2/2}\, \frac{1+\cos(4\pi x)}{2}, \quad \forall x \in \mathbb{T}, v \in [-v_*, v_*].$$



Diffusive limit (Fokker-Planck case)

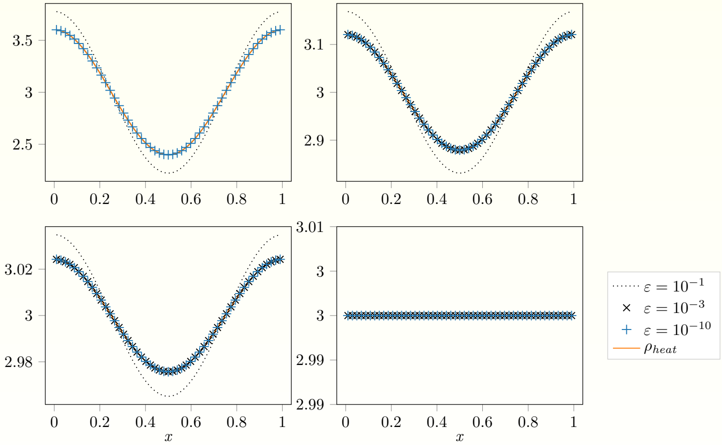


FIGURE – Comparison of the approximate solution of the heat equation (solid line) with the approximate densities obtained with the kinetic scheme for different ε, at times $t = 0.1$, 0.2, 0.3 and 10.

Diffusive limit (BGK case)

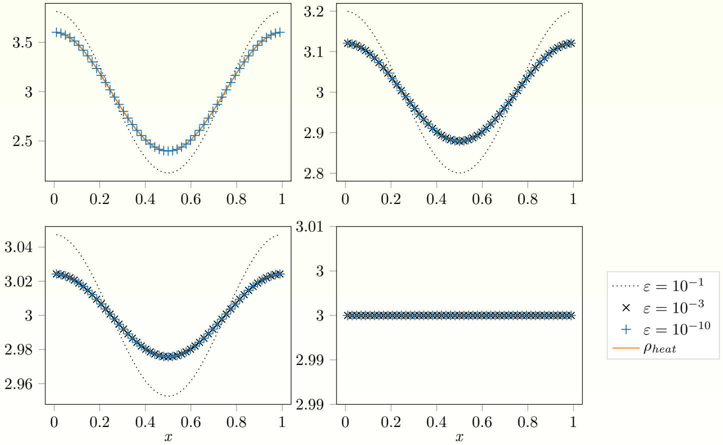


FIGURE – Comparison of the approximate solution of the heat equation (solid line) with the approximate densities obtained with the kinetic scheme for different ε, at times $t = 0.1$, 0.2, 0.3 and 10.

Trend to equilibrium

Fokker-Planck case, $\varepsilon = 1$.

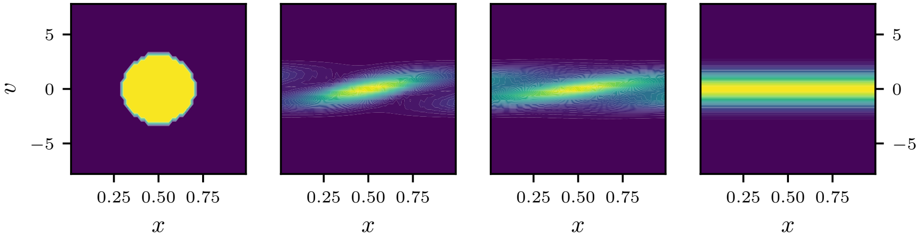


FIGURE – Snapshot of the particle distribution function $f^\varepsilon(t,x,v)$ in the $(x,v)-$phase-plane, at times $t = 0$, 0.3, 0.6, and 30.

Trend to equilibrium

Fokker-Planck case, $\varepsilon = 1$.

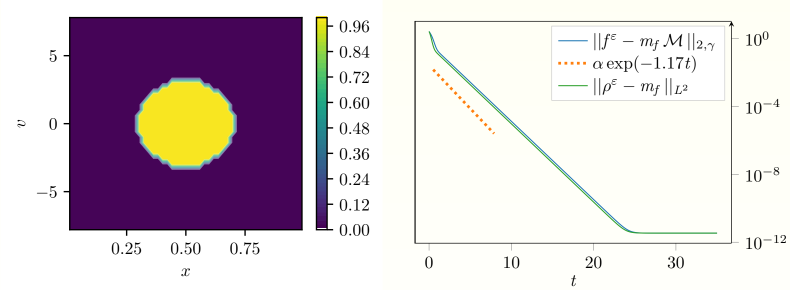


FIGURE – Left : Initial data in the $(x, v)-$phase-plane. Right : Time evolution of the weighted L^2 norm of the difference between f^ε and the global equilibrium.

Dependency of the hypocoercivity rate wrt ε

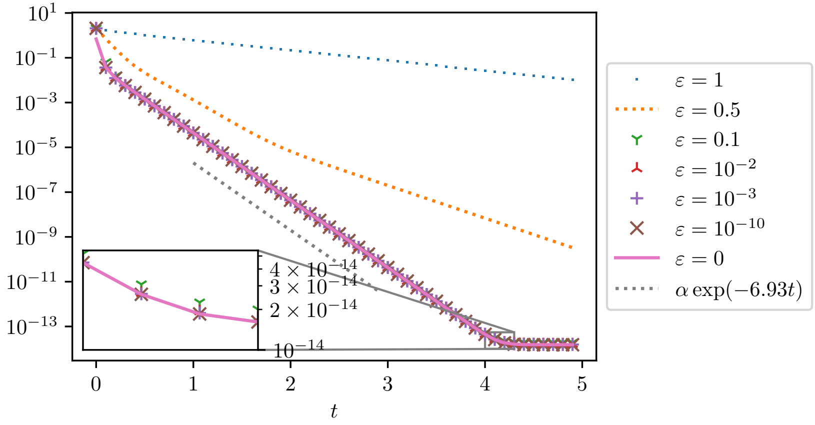


FIGURE – Comparison of the rate of convergence of $\|f - m_f \mathcal{M}\|_{2,\gamma}$ for different values of ε, in the **Fokker-Planck** case, with f_0 uniformly distributed in the $(x, v)-$phase-plane.

Conclusion

- Numerical scheme for a 1D linear kinetic equation.
- Proof of Asymptotic-Preserving property in the diffusion limit.
- Adaptation of hypocoercivity method to the discrete setting.
- Exponential return to equilibrium, uniformly in ε.

Outlook.

- Extension to more general linear collision operators ?
- Coupling with Poisson equation ?

Thank you for your attention !