Mixed Hodge structure of Open Fermat surfaces and $\underbrace{\text{hypergeometric function}}_{\text{(HGF)}}$

1. Motivation.
2. Outline of proof
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4. Explicit formula.

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1. Motivation

Hypergeometric function. (HGF)

$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n \, n!} z^n $$

defined for $\mathrm{Re}\, a_i, \mathrm{Re}\, b_i >> 0$, $|z| > 1$.

Continued analytically to $z = 1$.

By contiguity relation, defined $a_i, b_i \notin \mathbb{Z}$.

<u>Thm 1</u> (Asakura-Otsubo, AOT.)

$a, b, c \in \frac{1}{m}\mathbb{Z}$, m: minimal common denominator

Set $\alpha_1 = b + c - a$, $\alpha_2 = a - b$, $\alpha_3 = a - c$

$(\alpha_0 = -\alpha_1 - \alpha_2 - \alpha_3)$

If (NR) $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \notin \mathbb{Z}$

(HC) $\forall t \in (\mathbb{Z}/m\mathbb{Z})^\times$ $\langle t\alpha_0\rangle + \langle t\alpha_1\rangle + \langle t\alpha_2\rangle + \langle t\alpha_3\rangle = 2$

$$ \Longrightarrow {}_3F_2\left(\begin{matrix} 1, 1, a \\ b, c \end{matrix}; 1\right) \in \langle 1, \{\log a\}_{a \in \overline{\mathbb{Q}}} \rangle_{\overline{\mathbb{Q}}} $$

Rem. Condition (HC)

$\longleftrightarrow$ $H^2(X_m)(\alpha_1,\alpha_2,\alpha_3)$ is generated by algebraic cycles.

$\uparrow$ Fermat surface $\quad$ $\uparrow$ character

· For surface Hodge cycle $\longleftrightarrow$ Algebraic cycle
(Lefschetz - Hodge Theorem)

Def (Fermat surface) ($m \geqq 2$)

$$X_m : u^m + v^m = 1 + w^m$$

$$G_m = \mu_m \times \mu_m \times \mu_m \ni (\zeta_1, \zeta_2, \zeta_3)$$

$$\chi = (\alpha_1, \alpha_2, \alpha_3) \in \check{G}_m = \left(\frac{\mathbb{Z}}{m}/\mathbb{Z}\right)^{\oplus 3}$$

$$\chi : (\zeta_1, \zeta_2, \zeta_3) \longmapsto \zeta_1^{m\alpha_1} \cdot \zeta_2^{m\alpha_2} \zeta_3^{m\alpha_3} \in \mu_m$$

G_m acts on X_m

$$(u, v, w) \xrightarrow[(\zeta_1,\zeta_2,\zeta_3)]{} (\zeta_1 u, \zeta_2 v, \zeta_3 w)$$

$$H^2_B(X_m, \mathbb{Q}(\mu_m))(\chi) = \text{"The } \chi\text{-part of } H^2\text{"}$$

Prop (Shioda) $H^2_B(X_m, \mathbb{Q}(\zeta_m))(\chi)$ is generated by Hodge cycle $\Longleftrightarrow$ $(\alpha_1, \alpha_2, \alpha_3)$ satisfies (NR) + (HC).

2. Outline of proof of AO, AOT.

1^{st} proof AO

2^{nd} proof AOT

I : $(a_0, b_0, c_0) \in \mathbb{Q}^3$ with (NR)

By contiguity rel. $\left\{ {}_3F_2\left(\begin{matrix} k_1, k_2, a \\ b, c \end{matrix} \middle| 1 \right) \right\}$ $k_1, k_2 \in \mathbb{Z}$, $a \equiv a_0$, $b \equiv b_0$, $c \equiv c_0$ $(\mathbb{Z})$

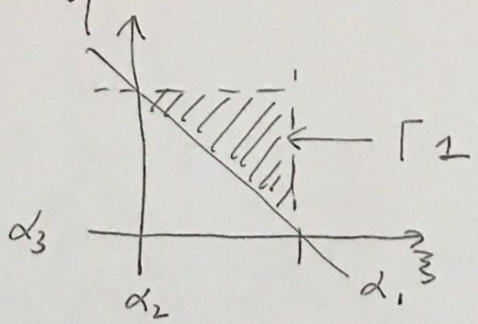
generates at most 2 dim vect space $/\overline{\mathbb{Q}}$.

II. We use formula

$$\int_{\Gamma_1} (\xi+\eta-1)^{\alpha_1-1}\, \xi^{\alpha_2-1}\, \eta^{\alpha_3-1}\, d\xi\, d\eta$$

$$= \frac{1}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)}\ {}_3F_2\left(\begin{matrix} 1, 1, \alpha_1+\alpha_2+\alpha_3 \\ \alpha_1+\alpha_2+1, \alpha_1+\alpha_3+1 \end{matrix}; 1\right)$$

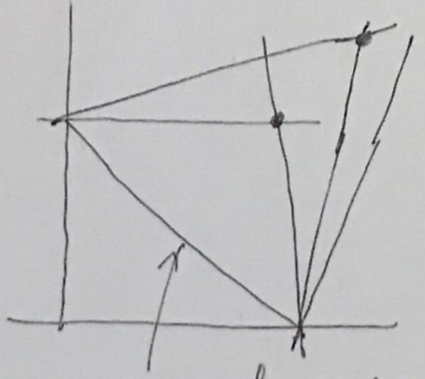
Right hand side is a period of relative cohomology of Fermat surf. Here Γ_1 is defined by

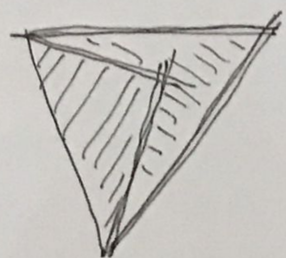


Right hand side: Set $\xi^{\frac{1}{m}} = u$, $\eta^{\frac{1}{m}} = v$, $(\xi+\eta-1)^{\frac{1}{m}} = w$ $(\Rightarrow u^m + v^m = 1 + w^m)$

$\Rightarrow$ diff form in RHS = diff form on X_m character $= (\alpha_1, \alpha_2, \alpha_3)$

$$B_m = \{\xi = 1\} \cup \{\eta = 1\}$$
$$= \{u^m = 1\} \cup \{v^m = 1\} = \bigcup_{\zeta\in\mu_m} \left(\{u = \zeta\} \cup \{v = \zeta\}\right)$$



$\widetilde{\Gamma}_1 =$ 

(union of 2 triangle)

$\partial \widetilde{\Gamma}_1 \subset B_m$

Integral RHS

$$= * \int_{\widetilde{\Gamma}_1} \omega^{m d_1 - 1} u^{m d_2} v^{m d_3} \frac{du}{u} \frac{dv}{v}$$

$\uparrow$ easy factor

$\cap$

$\Omega^2_{X_m^o}$, $X_m^o = X_m - \left(\begin{array}{l}\text{coordinate} \\ \text{hyperplane}\end{array}\right)$

$\cap$

$\mathbb{P}^3$

$(= {}_3F_2(1))$

We have mixed Hodge structures. coeff $= \mathbb{Q}(\mu_m)$
$= K$.

$$0 \to H^1_B(B_m)(\chi) \to H^2_B(X_m^o ; B_m)(\chi) \to H^2_B(X_m^o)(\chi) \to 0$$

$H^1_B(B_m)(\chi) \cong \mathbb{Q}(0)$

$H^2_B(X_m^o)(\chi) = \left(H^2_B(X_m)(\chi) \right) \underset{(2)}{\cong} \mathbb{Q}(1)$

◎ Lefschetz-Hodge thm

(2) is obtained from algebraic cycles.

$\therefore$ Extension class comes from $\mathrm{Ext}^1_{MM/\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}(1)) \otimes K$.

$\Rightarrow$ Thm.

<u>Q: Explicit expression?</u>

3. Aoki-Shioda algebraic cycles.

<u>Thm</u> ([AS]) $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Q}^3$, m = minimal common denom.

(NR) + (HC) $\Rightarrow$ classified (1)—(5) (mod $\mathbb{Z}$)

(1) $(\alpha, -\alpha, \beta, -\beta)$.

(2) $(\alpha, \alpha + \frac{1}{2}, -2\alpha, \frac{1}{2})$, $m = 2m'$

(3) $(\alpha, \alpha + \frac{1}{3}, \alpha + \frac{2}{3}, -3\alpha)$, $m = 3m'$

(4) $(\alpha + \frac{1}{4}, \alpha + \frac{3}{4}, 2\alpha, -4\alpha)$, $m = 4m'$

(5) $m \leq 180$ ($\Rightarrow$ finite set; exceptional case)

Main Theorem (T)

The case (1)—(4), we have explicit formula.

(Explain case (3)). We use.

Thm ([AS])

For (1) — (4), explicitly written algebraic cycles generate χ-part of $H^2_B(X_m^\circ)$.

(coefficient $= K$).

(1) $\Longrightarrow$ generated by class of lines.

Ex. (3) replace m' by m.

$$u^{3m} + v^{3m} = 1 + w^{3m}$$

set $u^m = x$, $v^m = y$, $w^m = z$.

$$0 = x^3 + y^3 - 1 - z^3 = (x^3 + y^3 - 1 + 3xy) + (-3xy - z^3)$$

$$= (x + y - 1)(x^2 + y^2 + 1 + xy - x - y) - (z + \sqrt[3]{3}xy)(z + \varepsilon\sqrt[3]{3}xy)(z + \varepsilon^2\sqrt[3]{3}xy)$$

$$\varepsilon^3 = 1.$$

$$\therefore Z_1 \begin{cases} x + y - 1 = u^m + v^m - 1 = 0 \\ w = \sqrt[3m]{3} \cdot * \, uv \end{cases}$$

$*$ root of 1.

$\Rightarrow Z_1 \subset X_m$.

G_m orbits of Z_1 generates $H^2(X_m^\circ)(\chi)$

4. Explicit expression.

Main Thm. case (3).

$$\Omega = \omega^{md_1} u^{md_2} v^{md_2} \frac{du}{u}\frac{dv}{v} \in \Omega^2_{X_m^\circ}(X)$$

Z = union of G_m orbit of $Z_1 \subset X$.

$$\underset{[AS]}{\Longrightarrow} \quad H^2_Z(X_m^\circ)(x) \longrightarrow H^2_Z(X_m^\circ)(x)$$

$$\Longrightarrow \quad H^2_Z(X_m^\circ)(x) \longrightarrow H^2_Z(X_m^\circ - Z)(x)$$

is zero

$$\therefore \exists \eta \in \Omega^1_{X_m^\circ - Z} \quad d\eta = \Omega|_{X_m^\circ - Z}.$$

We find η

Recall. $\partial \tilde{\Gamma}_1 \subset B$ moreover, we assume

① $B \cap Z$ = finite set

② $\tilde{\Gamma}_1 \cap Z = \phi$, i.e. $\tilde{\Gamma}_1 \subset X_m - Z$

Then $\displaystyle\int_{\tilde{\Gamma}_1} \Omega = \int_{\tilde{\Gamma}_1} d\eta = \int_{\partial\tilde{\Gamma}_1} \eta$ $\hookrightarrow$ This is explicitely calc..

Finding η

$$X_m : x^3 + y^3 = 1 + z^3. \qquad \begin{cases} x = u^m \\ y = v^m \\ z = \omega^m \end{cases}$$

$$\psi_0 := \sum \omega^{2i+j} d\log(\omega^i x + \omega^j y - 1)$$

$$f = \frac{(x^3+y^3-1)^3}{x^3y^3} \quad (\Rightarrow f|_{Z_1} = \text{const, recalled.})$$

set

$$\therefore \eta = f^\alpha \psi_0 \quad \left(\begin{array}{l}1\text{- form. residue is const.} \\ \text{and } x\text{-part}\end{array}\right)$$

$$\begin{tikzcd}
& \Omega^1_{X^\circ}(X) \arrow[r] \arrow[d] & \Omega^2_{X^\circ}(X) \arrow[d, hook] \\
\eta \arrow[ddd, mapsto] & \Omega^1_{X^\circ}(\log Z)(X) \arrow[r] \arrow[d, "\mathrm{Res}"'] & \Omega^2_{X^\circ}(\log Z)(X) \arrow[d] \\
& \mathcal{O}_{Z^\circ}(X) \arrow[r] & \Omega^1_{Z^\circ}(X) \\
\mathrm{Res}\,\eta \arrow[rr, mapsto] & & 0.
\end{tikzcd}$$

const residue.

$\exists$

$\therefore\ d\eta = \Omega'|_{X^\circ - Z}$

actually we have

$\Omega' = \text{const} \times \Omega.$

$\neq 0.$

have an explicit formula

<u>Thm.</u> $(\alpha_1, \alpha_2, \alpha_3) = (3\alpha - 1, -\alpha + \frac{1}{3}, -\alpha + \frac{2}{3})$

by computing $\displaystyle\int_{\Gamma_1} (\xi + \eta - 1)^{3\alpha - 1} \xi^{-\alpha - \frac{2}{3}} \eta^{-\alpha - \frac{1}{3}} d\xi d\eta$,

we have

$$\frac{\alpha}{(6\alpha + 1)(2\alpha + \frac{2}{3})} F\left(\begin{matrix} 1,\ 1,\ \alpha + 1 \\ 2\alpha + \frac{4}{3},\ 2\alpha + \frac{5}{3} \end{matrix} ; 1\right)$$

$$= \int_0^1 y^{6\alpha + 1} \frac{3 + y^3}{27 + y^6} dy + \int_0^1 x^{6\alpha} \frac{9 + x^3}{27 + x^6} dx$$

(by partial fraction $\in \langle 1, \{\log \alpha\}_{\alpha \in \overline{\mathbb{Q}}} \rangle_{\overline{\mathbb{Q}}}$.)