Single-Valued Multiple Zeta Values and String Amplitudes

Max-Planck-Institut für Physik
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Workshop "Amplitudes and Periods"
Hausdorff Research Institute for Mathematics
February 26 - March 2, 2018

based on:

- St.St.: **Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator,** J. Phys. A47 (2014) 155401, [arXiv:1310.3259]

- St.St., T.R. Taylor: **Closed string amplitudes as single-valued open string amplitudes,** Nucl. Phys. B881 (2014) 269-287, [arXiv:1401.1218]

- Wei Fan, A. Fotopoulos, St.St., T.R. Taylor: **Sv-map between Type I and Heterotic Sigma Models,** to appear in Nucl. Phys. B, [arXiv:1711.05821]

Outline

- Real iterated integral on $(\mathbf{RP}^1/\{0,1,\infty\})^{N-3}$

$$Z \sim \int_{x_1<\ldots<x_N} \left(\prod_{l=2}^{N-2} dx_l\right) \prod_{i<j} |x_i - x_j|^{\alpha' s_{ij}} \; (x_i - x_j)^{n_{ij}} \;, \quad s_{ij} \in \mathbf{R},\; n_{ij} \in \mathbf{Z}$$

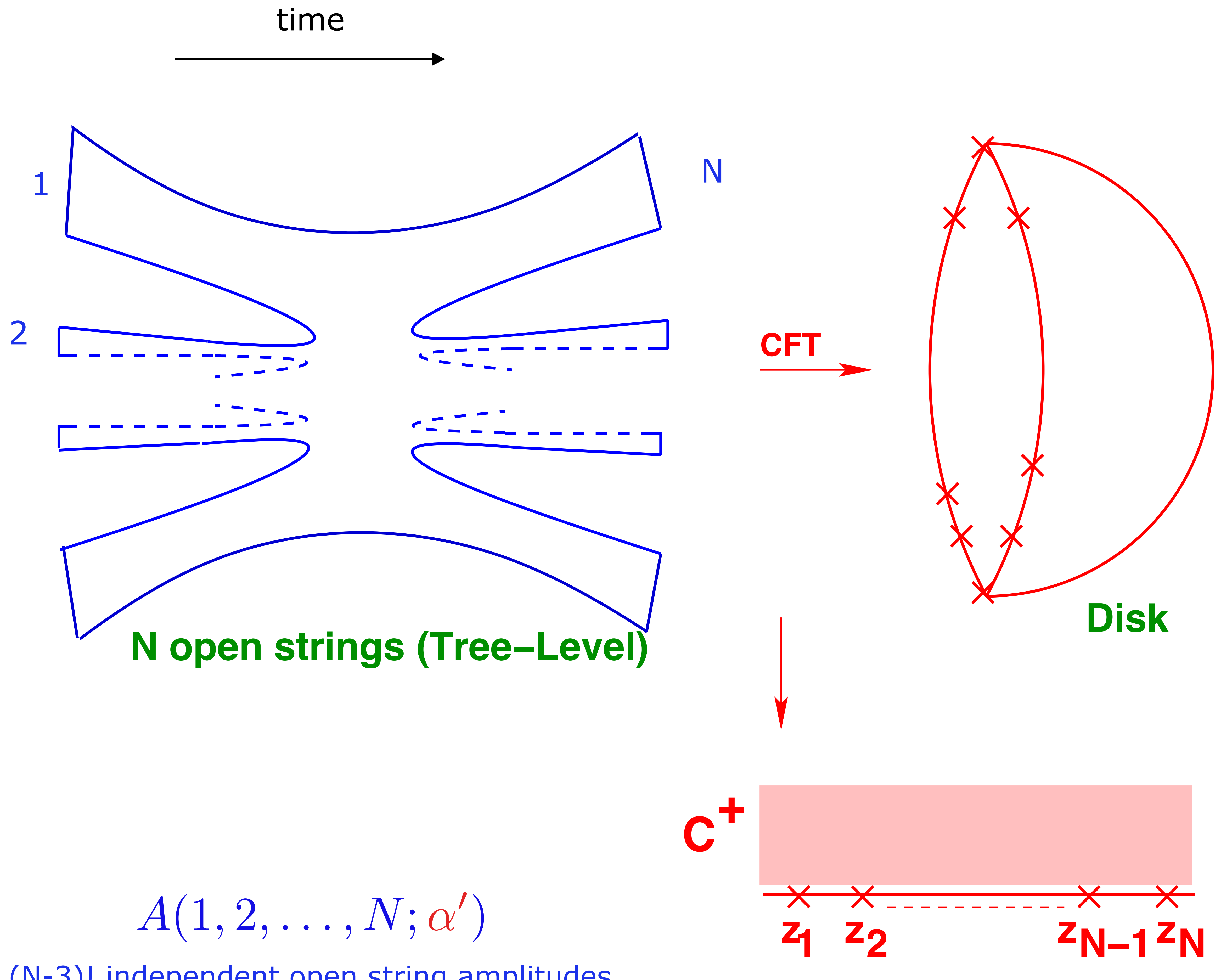
→ periods: MZVs
decomposition of motivic MZVs

- Complex integral on $(\mathbf{CP}^1/\{0,1,\infty\})^{N-3}$

$$J \sim \int_{\mathbf{C}^{N-3}} \left(\prod_{l=2}^{N-2} d^2 z_l\right) \prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}} \; (z_i - z_j)^{n_{ij}} \; (\overline{z}_i - \overline{z}_j)^{\overline{n}_{ij}}, \quad s_{ij} \in \mathbf{R},\; n_{ij}, \overline{n}_{ij} \in \mathbf{Z}$$

→ single-valued MZVs

- Relation: $J = \mathrm{sv}(Z)$ for given $\overline{n}_{ij}$


time
1
2
N
N open strings (Tree–Level)
CFT
Disk
C+
z_1 z_2 z_{N-1} z_N
$A(1, 2, \ldots, N; \alpha')$
(N-3)! independent open string amplitudes

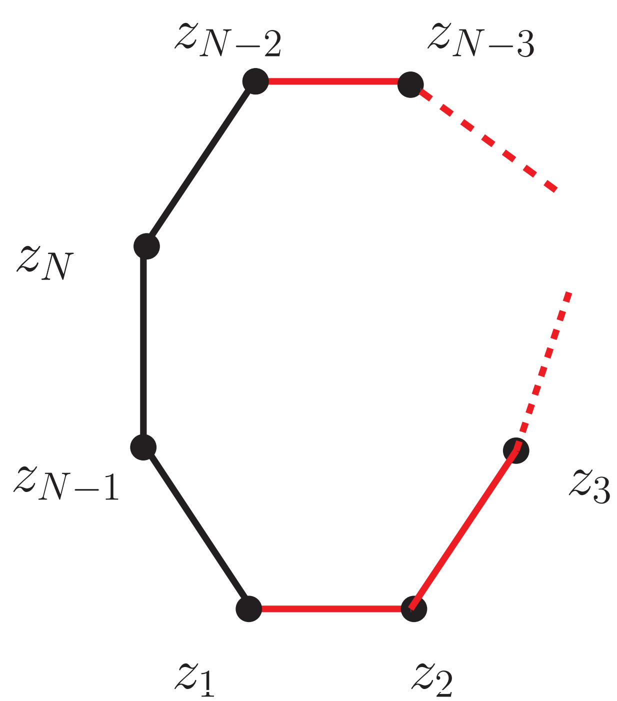
Actually we consider: $z_1 = 0, \quad z_{N-1} = 1, \quad z_N = \infty$ due to $PSL(2, \mathbf{R})$ symmetry

$$Z_\pi(\rho) := \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)} \; z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

iterated real integral on $(\mathbf{RP}^1/\{0,1,\infty\})^{N-3}$

$$\pi, \rho \in S_{N-3}$$

$$z_{ij} := z_i - z_j$$



$$D(\pi) = \left\{ \; z_j \in \mathbf{R} \mid 0 < z_{\pi(2)} < \ldots < z_{\pi(N-2)} < 1 \right\} \subset \left(\mathbf{RP}^1 \backslash \{0, 1, \infty\}\right)^{N-3}$$

Comments:

- Z = generalized Euler (Selberg) integral integrates to multiple Gaussian hypergeometric functions:
  Aomoto-Gelfand hypergeometric functions, GKZ structures

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- power series in α' :

Example:

$$\int\limits_{z_1<\ldots<z_5} \left(\prod_{l=2}^{3} dz_l\right) \frac{\prod\limits_{i<j}^{4} |z_{ij}|^{\alpha' s_{ij}}}{z_{12}\ z_{23}\ z_{41}}$$

$$= \alpha'^{-2} \left(\frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}}\right) + \zeta_2 \left(1 - \frac{s_{34}}{s_{12}} - \frac{s_{12}}{s_{45}} - \frac{s_{23}}{s_{45}} - \frac{s_{51}}{s_{23}}\right) + \mathcal{O}(\alpha'')$$

$$z_{ij} := z_i - z_j$$

* yields **iterated integrals**, which are **periods** of the moduli space $\mathcal{M}_{0,N}$ of genus zero curves with N ordered marked points

* integrate to Q-linear combinations of **MZVs** (Brown, Terasoma)

$$Z_\pi(\rho) := \int\limits_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod\limits_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\ z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

For given π all integrals can be expressed in $\mathbf{R}$ in terms of (N-3)! dimensional **basis**

For given ρ all integrals can be expressed in $\mathbf{R}$ in terms of (N-3)! dimensional **basis**

} fundamental world-sheet **disk** integrals

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normalization: $S^{-1} := (-1)^{N-3} \; Z\big|_{\alpha'^{3-N}}$

$$\boxed{F := Z\ S} \quad \text{i.e.: } F\big|_{\alpha'=0} = 1$$

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$F\ =\ (N-3)! \times (N-3)!$ matrix with $\mathrm{rk}(F) = (N-3)!$

$F\ =\ $ period matrix of $\mathcal{M}_{0,N}$

private discussion with S. Goncharov

S = KLT kernel

$$S[\rho|\sigma] \quad := \quad S[\,\rho(2,\ldots,N-2)\,|\,\sigma(2,\ldots,N-2)\,]$$

$$= \quad \prod_{j=2}^{N-2} \Big(s_{1,j_\rho} \; + \; \sum_{k=2}^{j-1} \theta(j_\rho, k_\rho)\, s_{j_\rho,k_\rho} \Big)$$

Bern, Dixon, Perelstein, Rozowsky (1998)

$$s_{ij} = \alpha'(k_i + k_j)^2$$

appears for gravitational amplitude
important for double copy constructions

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Physics: A_{YM} = $(N-3)!$ dimensional vector encompassing **all** independent **field-theory YM** subamplitudes $A_{YM}(\sigma),\ \sigma\in S_{N-3}$

$$A_{YM}(\sigma) = A_{YM}(1,\sigma(2,\ldots,N-2),N-1,N)$$

A = $(N-3)!$ dimensional vector encompassing **all** independent **superstring** subamplitudes $A(\sigma),\ \sigma\in S_{N-3}$

$$A = F\,A_{YM}$$

Mafra, Schlotterer, St.St. (2011)
Broedel, Schlotterer, St.St. (2013)

→ *F has also physical meaning*

Schlotterer, Stieberger, arXiv:1205.1516

Observation/Result

organization according to zeta values

$$P = 1 + \sum_{n\geq 1} \zeta_2^n \; P_{2n}, \quad P_{2n} = F(\alpha')|_{\zeta_2^n}$$

$$M_{2n+1} = F(\alpha')|_{\zeta_{2n+1}}$$

$$F(\alpha') \;=\; P \; Q \; \exp\left\{ \sum_{n\geq 1} \zeta_{2n+1} \; M_{2n+1} \right\}$$

$$Q \;=\; 1 + \tfrac{1}{5}\,\zeta_{3,5}\,[M_5, M_3] + \left\{ \tfrac{3}{14}\,\zeta_5^2 + \tfrac{1}{14}\,\zeta_{3,7} \right\}\,[M_7, M_3]$$
$$+ \;\left\{ 9\,\zeta_2\,\zeta_9 + \tfrac{6}{25}\,\zeta_2^2\,\zeta_7 - \tfrac{4}{35}\,\zeta_2^3\,\zeta_5 + \tfrac{1}{5}\,\zeta_{3,3,5} \right\}\,[M_3, [M_5, M_3]] + \ldots$$

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l}, \quad n_l \in \mathbf{N}^+,\; n_r \geq 2$$

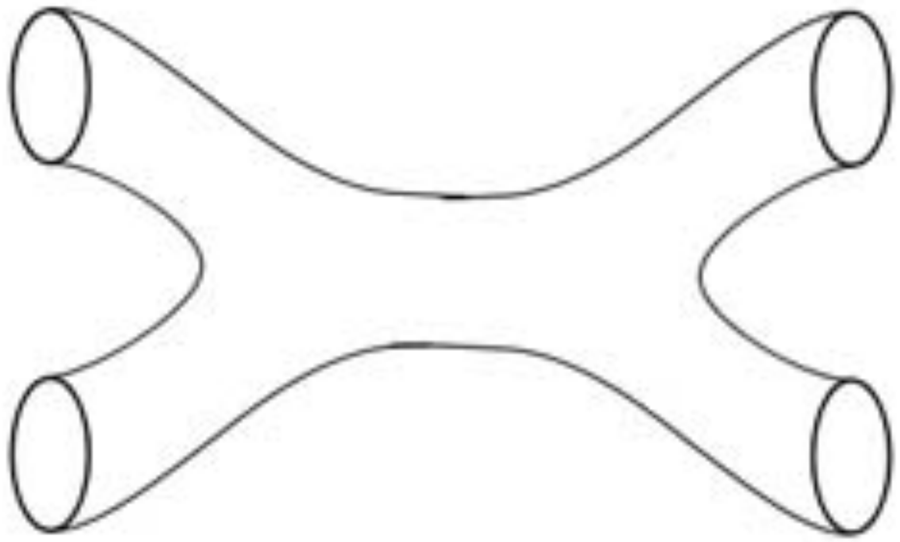
- all information is kept in P and M

E.g. N=5:

$$P_2 = {\alpha'}^2 \begin{pmatrix} -s_{34}\;s_{45} + s_{12}\;(s_{34} - s_{51}) & s_{13}\;s_{24} \\ s_{12}\;s_{34} & (s_{12} + s_{23})\;(s_{23} + s_{34}) - s_{45}\;s_{51} \end{pmatrix}$$

- this form exactly appears in F. Browns decomposition of motivic multiple zeta values
- coaction gives rise to factorisation of the amplitude

Construct **closed string amplitude**:
need a set of complex world-sheet integrals



N=4

Complex integral on $(\mathbf{CP}^1)^{N-3}$ (thrice punctered sphere)

$$J \sim \int_{\mathbf{C}} \left(\prod_{l=2}^{N-2} d^2 z_l \right) \prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}} \; (z_i - z_j)^{n_{ij}} \; (\overline{z}_i - \overline{z}_j)^{\overline{n}_{ij}}, \quad s_{ij} \in \mathbf{R}, \; n_{ij}, \overline{n}_{ij} \in \mathbf{Z}$$

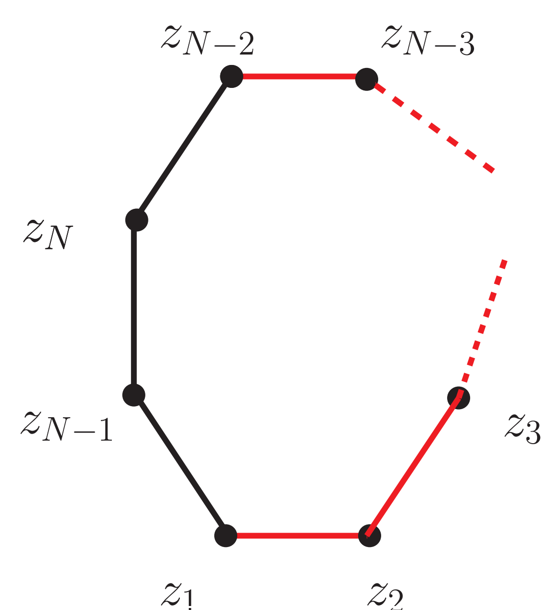
Real iterated integrals vs. complex integrals

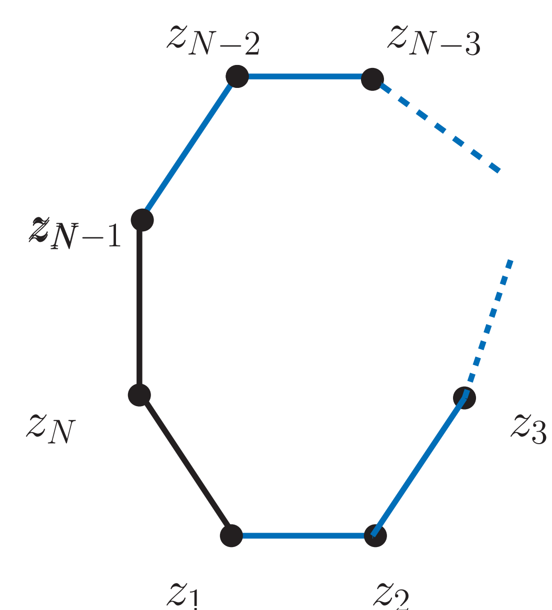
Recall: we considered the real iterated integral $\pi, \rho, \overline{\rho} \in S_{N-3}$

$$Z_\pi(\rho) := \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\; z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

$$D(\pi) = \{\; z_j \in \mathbf{R} \;\mid\; 0 < z_{\pi(2)} < \ldots < z_{\pi(N-2)} < 1 \;\}$$
$$\subset (\mathbf{RP^1}/0,1,\infty)^{N-3}$$

$$J(\rho,\overline{\rho}) := \int_{\mathbf{C}^{N-3}} \left(\prod_{j=2}^{N-2} d^2 z_j \right) \frac{\prod_{i<j}^{N-1} |z_{ij}|^{\alpha' s_{ij}}}{z_{1,\rho(2)}\; z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}} \; \frac{1}{\overline{z}_{1,\overline{\rho}(2)}\; \overline{z}_{\overline{\rho}(2),\overline{\rho}(3)} \cdots \overline{z}_{\overline{\rho}(N-2),N-1}}$$





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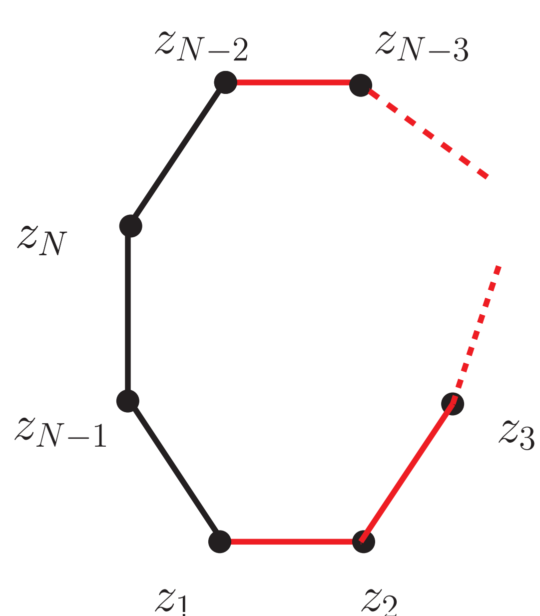
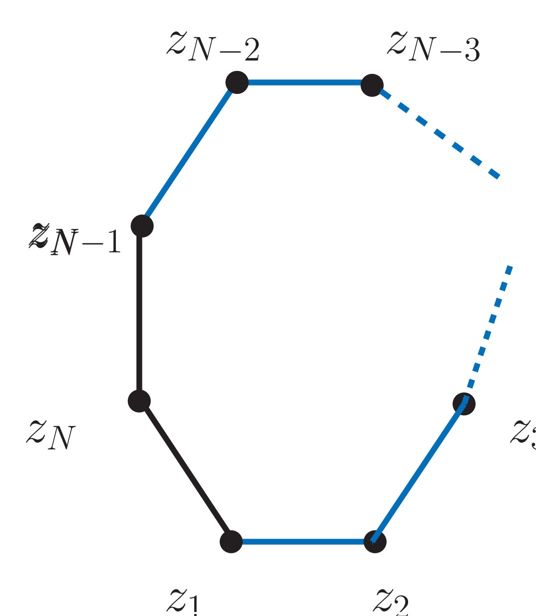
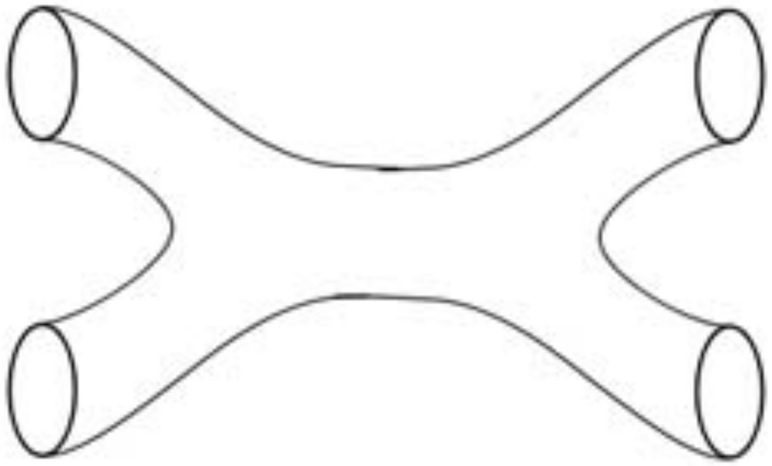


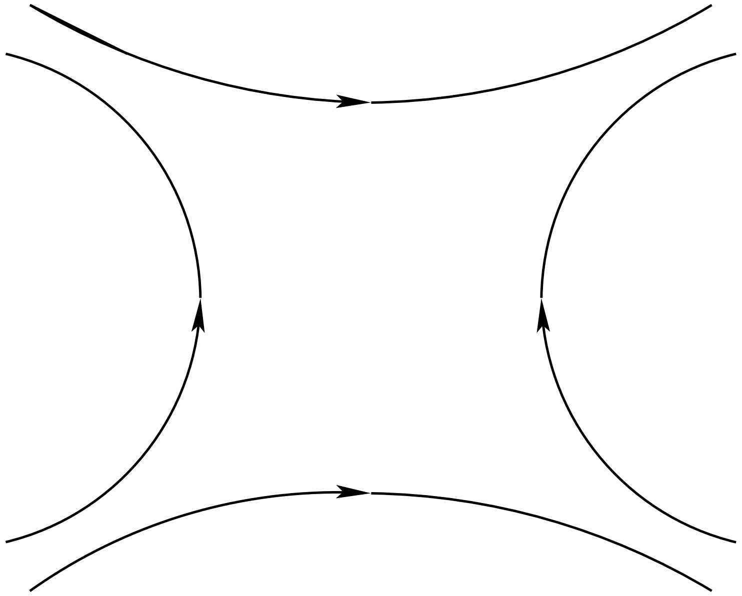


$$J = \text{sv}\,(Z)$$

e.g. N=4:



$= \mathrm{sv}$



$$\int_{\mathbf{C}} d^2z \; \frac{|z|^{2s} \; |1-z|^{2u}}{z \; (1-z) \; \overline{z}} = \mathrm{sv} \left(\int_0^1 dx \; x^{s-1} \; (1-x)^u \right)$$

complex integral on $(\mathbf{CP}^1/\{0,1,\infty\})^{N-3}$ iterated real integral on $(\mathbf{RP}^1/\{0,1,\infty\})^{N-3}$

$$\frac{1}{s} \; \frac{\Gamma(s) \; \Gamma(u) \; \Gamma(t)}{\Gamma(-s) \; \Gamma(-u) \; \Gamma(-t)} = \mathrm{sv} \left(\frac{\Gamma(s) \; \Gamma(1+u)}{\Gamma(1+s+u)} \right)$$

$$s = \alpha'(k_1+k_2)^2$$
$$t = \alpha'(k_1+k_3)^2$$
$$u = \alpha'(k_1+k_4)^2$$

$$\frac{1}{s} - 2 \; u \; t \; \zeta_3 + \mathcal{O}(\alpha'^4)$$

$$\frac{1}{s} - u \; \zeta_2 - u \; t \; \zeta_3 - \frac{1}{10} \; u \; (4s^2 + su + 4u^2) \; \zeta_2^2 + \mathcal{O}(\alpha'^4)$$

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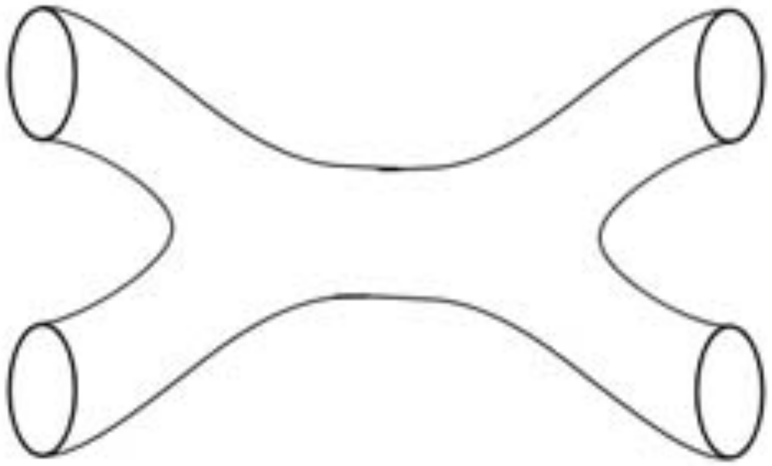

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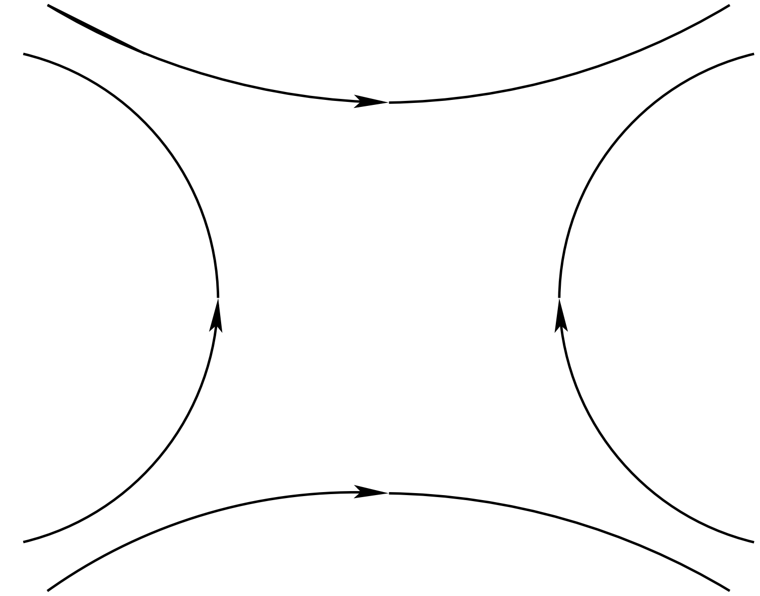

$$\int_{\mathbf{C}} d^2 z \; \frac{|z|^{2s} \; |1-z|^{2u}}{z \; (1-z) \; \overline{z}} = \text{sv} \left(\int_0^1 dx \; x^{s-1} \; (1-x)^u \right)$$

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$$\frac{1}{s} - u \; \zeta_2 \; 2 u \; t \; \zeta_3 - \frac{1}{10} \; u \; (4s^2 + su + 4u^2) \; \zeta_2^2 + \mathcal{O}(\alpha'^4)$$

Complex vs. iterated integrals

N=5:

$$\begin{pmatrix} \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} & \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{12}\overline{z}_{23}\overline{z}_{34}} \\ \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{12}z_{23}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} & \displaystyle\int_{z_2,z_3\in\mathbf{C}} d^2z_2\ d^2z_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{13}z_{32}\ \overline{z}_{13}\overline{z}_{32}\overline{z}_{24}} \end{pmatrix}$$

$$= \mathrm{sv}\begin{pmatrix} \displaystyle\int_{0<z_2<z_3<1} dz_2\ dz_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int_{0<z_2<z_3<1} dz_2\ dz_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \\ \displaystyle\int_{0<z_3<z_2<1} dz_2\ dz_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{12}z_{23}} & \displaystyle\int_{0<z_3<z_2<1} dz_2\ dz_3\ \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{13}z_{32}} \end{pmatrix}$$

Physics: "double copy constructions"

$$\tilde{A}_{YM}(\rho) = A_{YM}(1, \rho(2, \ldots, N-2), N, N-1)$$

$$\mathcal{M}_{FT} = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} \tilde{A}_{YM}(\rho)\ S[\rho|\sigma]\ A_{YM}(\sigma)$$

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$$A^{\mathrm{I}}(\pi) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} Z_{\pi}(\rho)\, S[\rho|\sigma]\, A_{YM}(\sigma)$$

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$$\mathcal{A}^{\mathrm{HET}}(\Pi) = \mathrm{sv}\left(\mathcal{A}^{\mathrm{I}}(\Pi)\right)$$

$$J[\pi|\rho] = \mathrm{sv}(Z_\pi(\rho))$$

$$A^{\mathrm{HET}}(\pi) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} J[\pi|\rho]\; S[\rho|\sigma]\; A_{YM}(\sigma)$$

Taylor, St.St. (2014)

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$$\mathcal{M}_N = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} \tilde{A}^{\mathrm{HET}}(\rho)\ S[\rho|\sigma]\ A_{YM}(\sigma)$$

Taylor, St.St. (2014)

F. Brown (2013):

SVMZVs are coefficients of the associator W

Deligne introduced associator W formally as:

$$W \circ {}^{\sigma}Z = Z$$

with Ihara action $\circ$ providing formal multiplication rule on group-like formal power series in e_0 and e_1

$$F(e_0, e_1) \circ G(e_0, e_1) = G(e_0, F(e_0, e_1)e_1 F(e_0, e_1)^{-1})\ F(e_0, e_1)$$

$$\Longrightarrow \quad W(e_0, e_1) = {}^{\sigma}Z(e_0, We_1W^{-1})^{-1}\ Z(e_0, e_1)$$

(definition only uses Ihara action)

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(definition only uses Ihara action)

Drinfeld associator Z:

$$Z(e_0,e_1) = \sum_{w\in\{e_o,e_1\}^\times} \zeta(w)\ w = 1 + \zeta_2\ [e_0,e_1] + \zeta_3\ (\ [e_0,[e_0,e_1]]] - [e_1,[e_0,e_1]\ ) + $$

$$\zeta(e_1e_0^{n_1-1}\ldots e_1e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r}$$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \sqcup\!\sqcup\, w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

Deligne associator W:

$$W(e_0,e_1) = Z(-e_0,-e_1')^{-1}\ Z(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta_{\rm sv}(w)\ w$$

$$W(e_0,e_1) = 1 + 2\ \zeta_3\ (\ [e_0,[e_0,e_1]]] - [e_1,[e_0,e_1]\ ) + \ldots$$

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$$W \circ {}^{\sigma}Z = Z$$

with Ihara action $\circ$ providing formal multiplication rule on group-like formal power series in e_0 and e_1

$$F(e_0,e_1) \circ G(e_0,e_1) = G(e_0, F(e_0,e_1)e_1F(e_0,e_1)^{-1})\ F(e_0,e_1)$$

$$\Longrightarrow \quad W(e_0,e_1) = {}^{\sigma}Z(e_0, We_1W^{-1})^{-1}\ Z(e_0,e_1)$$

(definition only uses Ihara action)

Drinfeld associator Z:

$$Z(e_0,e_1) = \sum_{w\in\{e_o,e_1\}^{\times}} \zeta(w)\ w = 1 + \zeta_2\ [e_0,e_1] + \zeta_3\ (\ [e_0,[e_0,e_1]]] - [e_1,[e_0,e_1]\ ) + \ldots$$

$$\zeta(e_1e_0^{n_1-1}\ldots e_1e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r}$$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \sqcup\!\sqcup\, w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

Deligne associator W:

$$W(e_0,e_1) = Z(-e_0,-e_1')^{-1}\ Z(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^{\times}} \zeta_{\rm sv}(w)\ w$$

$$W(e_0,e_1) = 1 + 2\ \zeta_3\ (\ [e_0,[e_0,e_1]]] - [e_1,[e_0,e_1]\ ) + \ldots$$

F. Brown (2013):

there is a natural homomorphism:

$$\mathrm{sv}: \; \mathcal{P}^a \xrightarrow{\mathrm{sv}^m} \mathcal{P}^m \xrightarrow{\mathrm{per}} \mathbf{C}$$

$\mathcal{P}^a$: unipotent de Rahm MZV's $\zeta^a \in \mathcal{A}$ (Goncharov)

$\mathcal{P}^m$: motivic MZV's $\zeta^m \in \mathcal{H}$ (Brown)

$$\mathrm{sv}: \zeta_{n_1,\ldots,n_r} \longrightarrow \zeta_{\mathrm{sv}}(n_1,\ldots,n_r)$$

E.g.:

$$\begin{aligned} \zeta_{\mathrm{sv}}(2) &= 0 \\ \zeta_{\mathrm{sv}}(2n+1) &= 2\,\zeta_{2n+1} \\ \zeta_{\mathrm{sv}}(3,5) &= -10\,\zeta_3\,\zeta_5 \end{aligned}$$

Note: explicit representation of associators in limit $\mathrm{mod}\ (\mathrm{g}')^2$

(corresponds to a commutative realization of the Ihara bracket)

$(g')^2 = [g,g]^2$

$u = -\mathrm{ad}_{e_1},\ v = \mathrm{ad}_{e_0}$

$\mathrm{ad}_x y = [x,y]$

$$Z(e_0,e_1) = 1 - (uv)^{-1} \left(\frac{\Gamma(1-u)\ \Gamma(1-v)}{\Gamma(1-u-v)} - 1 \right) [e_0,e_1]$$

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relates to open superstring amplitude

Drummond, Ragoucy (2013)

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relates to open superstring amplitude

Drummond, Ragoucy (2013)

$$W(e_0,e_1) = 1 + (uv)^{-1} \left(\frac{\Gamma(-u)\ \Gamma(-v)\ \Gamma(u+v)}{\Gamma(u)\ \Gamma(v)\ \Gamma(-u-v)} + 1 \right) [e_0,e_1]$$

relates to closed superstring amplitude

St.St. (2013)

Sv-map
between type I and heterotic sigma models

sv-map can also be anticipated at the **D=2 sigma models**
describing the world-sheet couplings of open and heterotic strings
mapping respective interaction terms (or **individual Feynman diagrams**)

$$S^I = \int d\tau \ g \ \left\{ A_\mu(X) \ \partial_\tau X^\mu - \frac{1}{2} \psi^\mu \psi^\nu \ F_{\mu\nu} \right\} + \ldots$$

background $F_l = F_l(F, D)$

$$\beta^I = \sum_{l \geq 1} F_l \ I_l^I \Big|_{\ln \epsilon}$$

beta-function corresponding to
boundary coupling $A_\mu \partial_\tau X^\mu$

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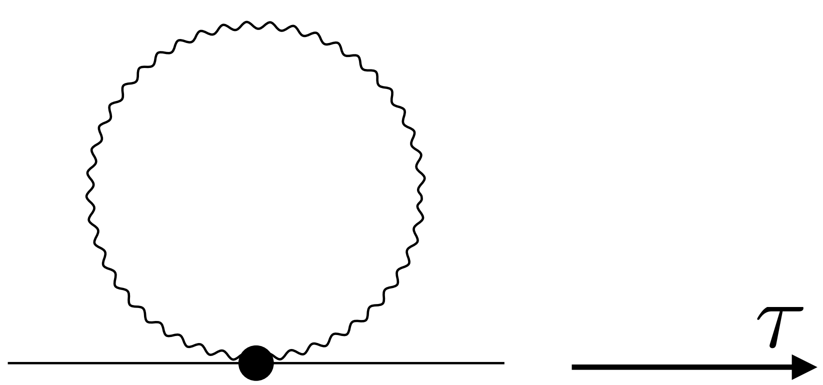
perturbation theory in D=2:

background $F_l = F_l(F, D)$

$$\beta^I = \sum_{l \geq 1} F_l \ I_l^I \Big|_{\ln \epsilon} = \alpha' \ g \ DF + \mathcal{O}({\alpha'}^2)$$

beta-function corresponding to boundary coupling $A_\mu \partial_\tau X^\mu$

simplest case l=1:



$A_\mu \partial_\tau X^\mu$

$$\Delta S^I = \int d\tau \ D_\nu F_{\mu\lambda} \ \partial_\tau X^\mu \ G^{\nu\lambda}(\tau, \tau') \Big|_{\tau \to \tau'}$$

$$G^{\nu\lambda}(\tau, \tau') = -\delta^{\nu\lambda} \ \ln |\tau - \tau'|$$

type I open string sigma-model

$$S^I = \int d\tau \; g \; \left\{ A_\mu(X)\; \partial_\tau X^\mu - \frac{1}{2}\psi^\mu \psi^\nu \; F_{\mu\nu} \right\} + \ldots$$

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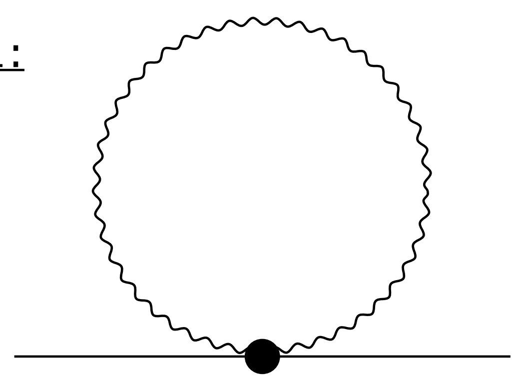

$A_\mu \partial_\tau X^\mu$

type I open string sigma-model	closed heterotic string sigma-model
$$S^I = \int d\tau \; g \; \left\{ A_\mu(X)\; \partial_\tau X^\mu - \frac{1}{2}\psi^\mu \psi^\nu \; F_{\mu\nu} \right\} + \ldots$$	Actually the heterotic sigma-model action can be **reorganized** and mapped onto open string one: $$S^{HET} = \int d^2z \; g \; \Psi \left\{ A_\mu(X)\; \partial_z X^\mu - \frac{1}{2}\psi^\mu \psi^\nu \; F_{\mu\nu} \right\} \Psi + \ldots$$

$$\beta^I = \sum_{l \geq 1} \; F_l \; I_l^I \Big|_{\ln \epsilon}$$

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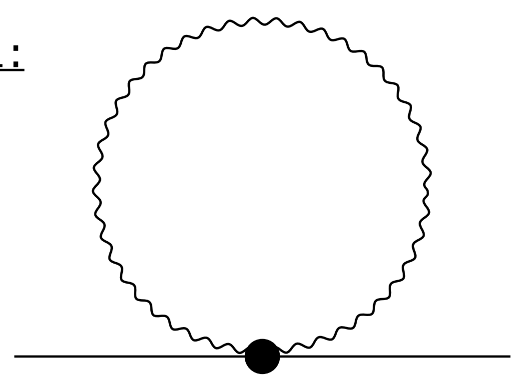

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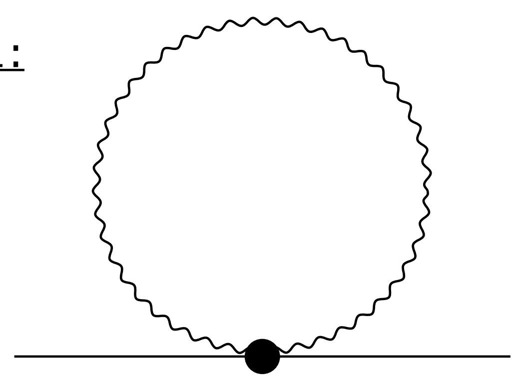
After this **reorganization** the heterotic string perturbation expansion is **diagrammatically equivalent** to the open string case

$$\beta^I = \sum_{l \geq 1} F_l \; I_l^I \Big|_{\ln \epsilon}$$

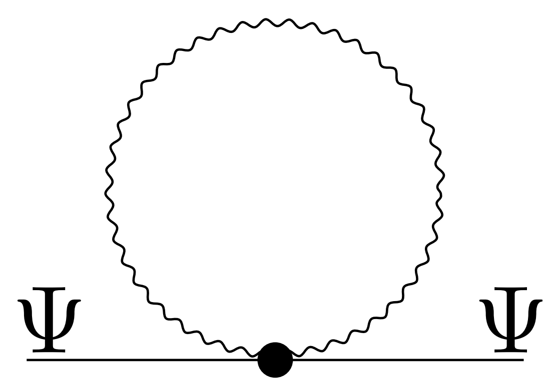
simplest case l=1:



$A_\mu \partial_\tau X^\mu$

$$\beta^{HET} = \sum_{l \geq 1} F_l \; I_l^{HET} \Big|_{\ln \epsilon}$$

simplest case l=1:



$A_\mu \partial_z X^\mu$

type I open string sigma-model	closed heterotic string sigma-model
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After this **reorganization** the heterotic string perturbation expansion is **diagrammatically equivalent** to the open string case

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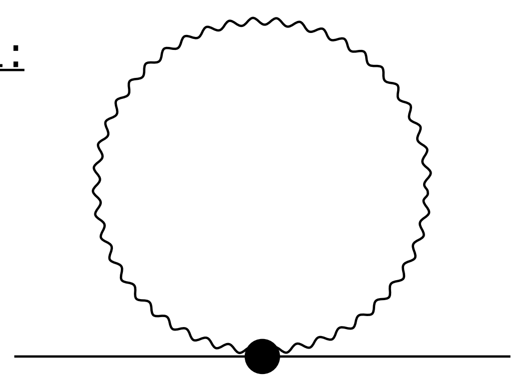

$A_\mu \partial_\tau X^\mu$

$$\beta^{HET} = \sum_{l\geq 1} F_l \ I_l^{HET} \Big|_{\ln \epsilon}$$

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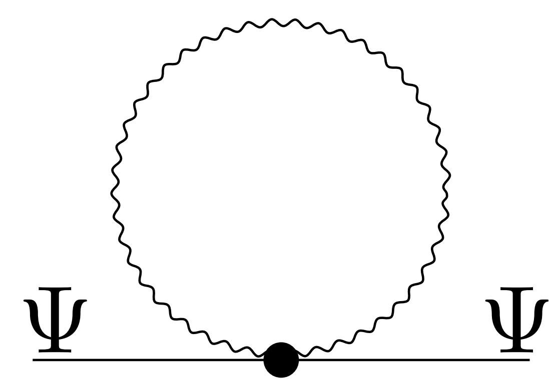
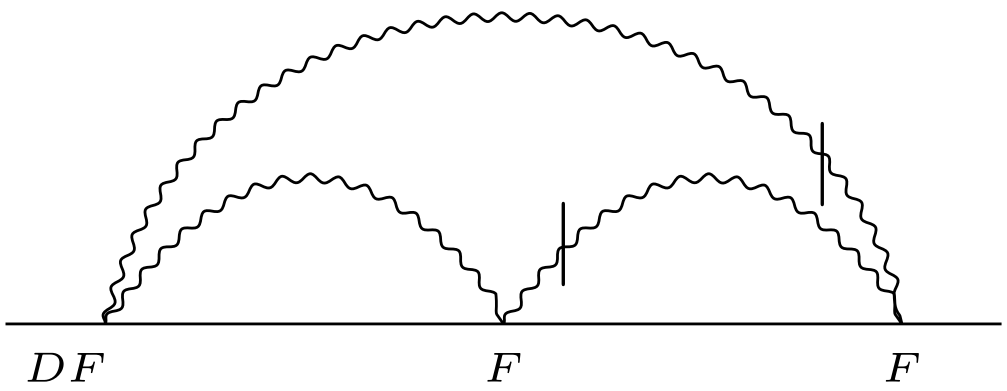


$A_\mu \partial_z X^\mu$

Proposal: $\beta^{HET} = \mathrm{sv}(\beta^I)$

$$(DF)F^{l-1} \hookrightarrow F^{l+1}$$

three-loop $l = 3$



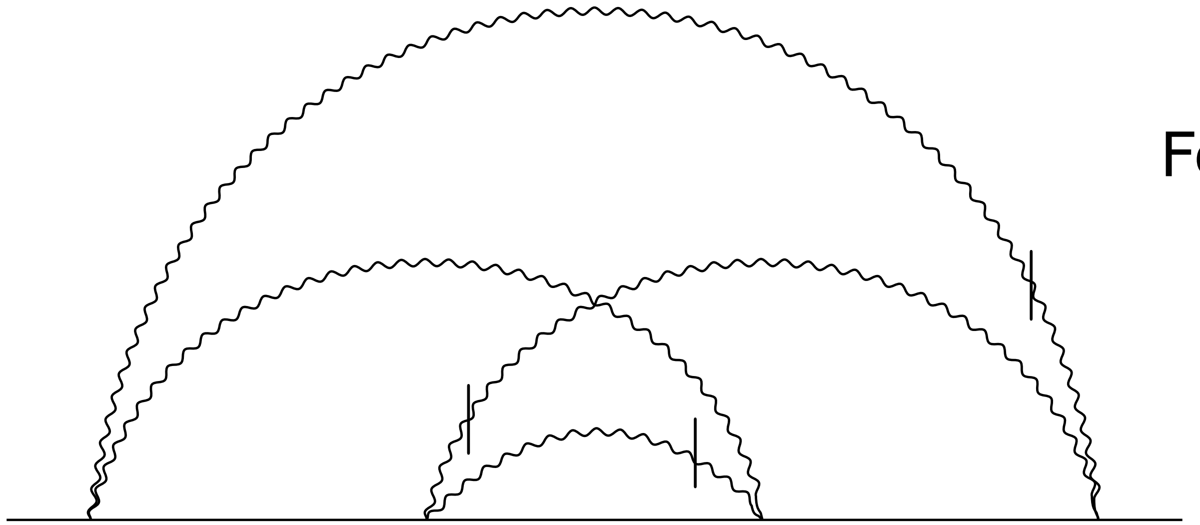
Feynman diagram corresponding to structure

$$\partial X^{\nu} D_{\mu_1} F_{\nu}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_1} \quad \hookrightarrow \quad F^4$$

$$I_3^I = \int_{-\infty<t_1<t_2<t_3<\infty} dt_1 dt_2 dt_3 \; \frac{\ln t_{21}}{t_{31} t_{32}} = \int_{-\infty}^{\infty} dt_1 \; \zeta_2 \; \ln \epsilon + \dots$$

$$I_3^{HET} = \int_{\mathbf{C}} \prod_{j=1}^{3} d^2 z_j \; \frac{1}{\bar{z}_{12} \bar{z}_{23}} \; \frac{\ln |z_{12}|^2}{z_{23} z_{13}} = \int d^2 z_1 \times 0 \times \ln \epsilon + \dots$$

four-loop $l = 4$



Feynman diagram corresponding to structure

$$\partial X^{\nu} D_{\mu_1} F_{\nu}{}^{\mu_3} F_{\mu_4}{}^{\mu_5} F_{\mu_3}{}^{\mu_4} F_{\mu_5}{}^{\mu_1} \hookrightarrow F^5$$

$$\int_{-\infty<t_1<t_2<t_3<t_4<\infty} dt_1 dt_2 dt_3 dt_4 \; \frac{\ln t_{31}}{t_{41} t_{42} t_{32}} = \int_{-\infty}^{\infty} dt_1 \; \zeta_3 \ln \epsilon + \ldots$$

$\updownarrow$

$$\int_{\mathbf{C}} \prod_{j=1}^{j=4} d^2 z_j \frac{1}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{34}} \; \frac{\ln |z_{13}|^2}{z_{14} z_{24} z_{23}} = \int d^2 z_1 \; 2\,\zeta_3 \ln \epsilon + \ldots$$

Remarks

- sv-map acts on individual graphs
  (associated to individual terms in effective action)

- reformulated perturbation theory
  in terms of identical Feynman diagrams

- proposal relies on a “sv-compatible regularization scheme”

  (tangential base point regularization)

Addon: coefficients of an associator W:

(reduced) KZ equation: $$\frac{d}{dz}\, L_{e_0,e_1}(z) = L_{e_0,e_1}(z)\, \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right)$$ with generators e_0 and e_1 of the free Lie algebra g

its unique solution can be given as generating series of multiple polylogarithms:

$$L_{e_0,e_1}(z) = \sum_{w\in\{e_0,e_1\}^\times} L_w(z)\; w$$

with the symbol $w \in \{e_0, e_1\}^\times$ denoting a non-commutative word $w_1 w_2 \ldots$ in the letters $w_i \in \{e_0, e_1\}$

$$L_1 = 1, \quad L_{e_0^n} = \frac{1}{n!}\ln^n z, \quad L_{e_1^n} = \frac{1}{n!}\ln^n(1-z)$$

Drinfeld associator Z:

$$\zeta(e_1 e_0^{n_1-1}\ldots e_1 e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r}$$

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \sqcup\!\sqcup\, w_2) \text{ and } \zeta(e_0) = 0 = \zeta(e_1)$$

$$Z(e_0,e_1) := L_{e_0,e_1}(1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta(w)\; w = 1 + \zeta_2\,[e_0,e_1] + \zeta_3\,(\,[e_0,[e_0,e_1]] - [e_1,[e_0,e_1]\,]\,) + \ldots$$

F. Brown (2004) defines generating series of SVMPs:

$$\mathcal{L}_{e_0,e_1}(z) = L_{-e_0,-e_1'}(\overline{z})^{-1}\; L_{e_0,e_1}(z)$$

e_1' determined recursively by fixed–point equation:

$$Z(-e_0,-e_1')\; e_1'\; Z(-e_0,-e_1')^{-1} = Z(e_0,e_1)\; e_1\; Z(e_0,e_1)^{-1}$$

Deligne associator W:

$$W(e_0,e_1) := \mathcal{L}(1) = Z(-e_0,-e_1')^{-1}\; Z(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^\times} \zeta_{\rm sv}(w)\; w$$

$$W(e_0,e_1) = 1 + 2\,\zeta_3\,([e_0,[e_0,e_1]] - [e_1,[e_0,e_1]]) + \ldots$$

F. Brown (2013)