Multivalued Feynman Graphs

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Motivation

- What type of multi-valued function does a Feynman graph generate?
- What is the role of graph complexes here?
- Understand Fubini, iterated Feynman integrals and general sheet structure.
- Cutkosky Rules
  How often can we cut, what do we learn?

Literature

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A cell complex for graphs: Outer Space

Useful concepts for the study of amplitudes:

- Outer Space itself as a cell-complex with a corresponding spine and partial order defined from shrinking edges;
- a cubical chain complex resulting from a boundary d which acts on pairs (Γ, F), F a spanning forest of Γ,
- a bordification which blows up missing cells at infinity.

The use of metric graphs suggests itself in the study of amplitudes upon using the parametric representation: the parametric integral is then the integral over the volume of the open simplex σ_Γ assigned to Γ in Outer Space.
Coloured edges reflect the possibility of different masses in the propagators assigned to edges. External edges are not drawn. Momentum conservation allows to incorporate them by connecting external vertices to a distinguished vertex v_∞.

Cutkosky's theorem

No loops formed by edges $\in E'$, else Fubini, then:

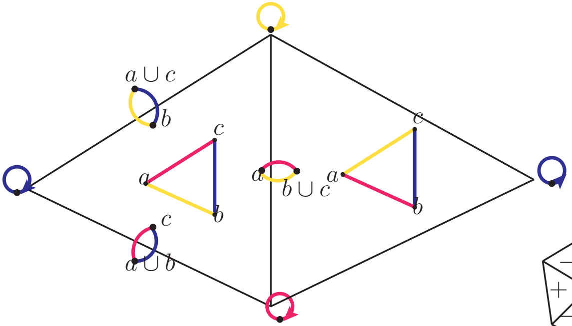
Theorem (Cutkosky)

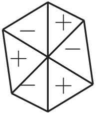
Assume the quotient graph G'' has a physical singularity at an external momentum point $p'' \in (\bigoplus_{V''} \mathbb{R}^D)^0$, i.e. the intersection $\bigcap_{e \in E''} Q_e$ of the propagator quadrics associated to edges in E'' has such a singularity at a point lying over p''. Let $p \in (\bigoplus_V \mathbb{R}^D)^0$ be an external momentum point for G lying over p''. Then the variation of the amplitude $I(G)$ around p is given by Cutkosky's formula

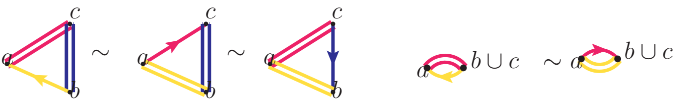
$$var(I(G)) = (-2\pi i)^{\# E''} \int \frac{\prod_{e \in E''} \delta^+(\ell_e)}{\prod_{e \in E'} \ell_e}. \quad (1)$$

The core co-product on graphs gives $m(\Phi_R \otimes \Phi_{CC} P)\Delta_c$ which allows to reduce the general case to the desired case.

Two triangular cells for the triangle graph:



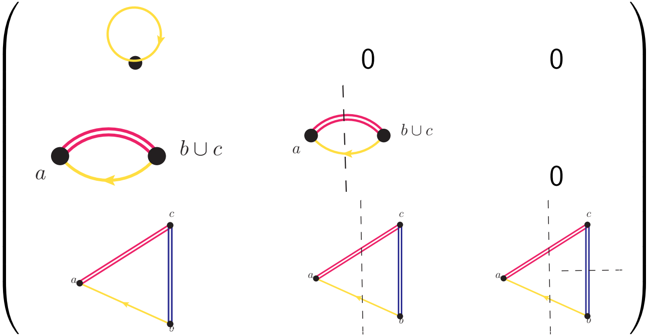


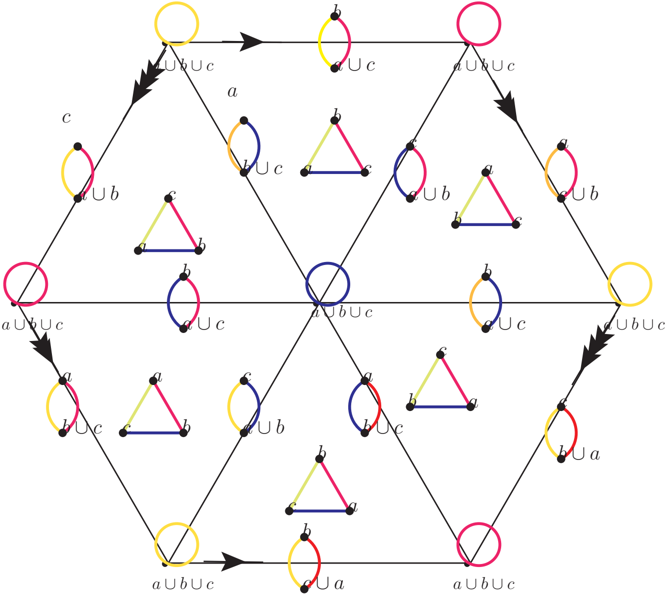


Exchange of yellow and red edges equals an orientation change for the loop!

$$\begin{pmatrix} 1 & 0 & 0 \\ (B_2(s;m_r^2,m_y^2) - B_2(m_y^2;m_r^2,m_y^2)) & V_{ry}(s,m_r^2,m_y^2) & 0 \\ \Phi_R(\Delta)(s,p_1^2,p_2^2;m_r^2,m_y^2,m_b^2) & \frac{1}{\sqrt{s,p_1^2,p_2^2}}\ln\frac{a+b}{a-b} & \frac{1}{\sqrt{s,p_1^2,p_2^2}} \end{pmatrix}$$

$\sim$





a ∪ b ∪ c
b
a ∪ c
a ∪ b ∪ c
a
c
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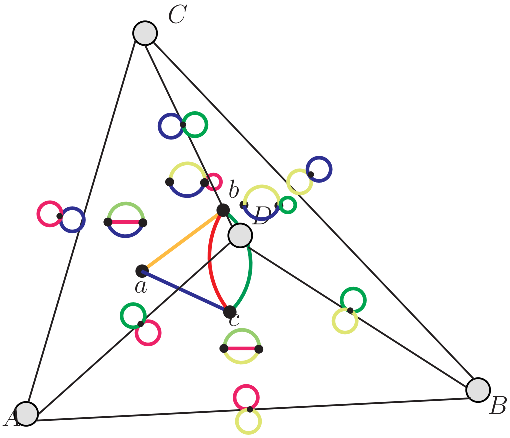
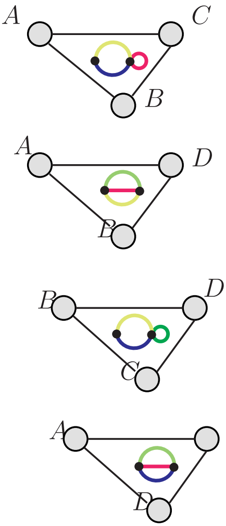
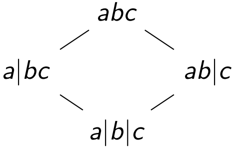
Second example: the Dunce's cap



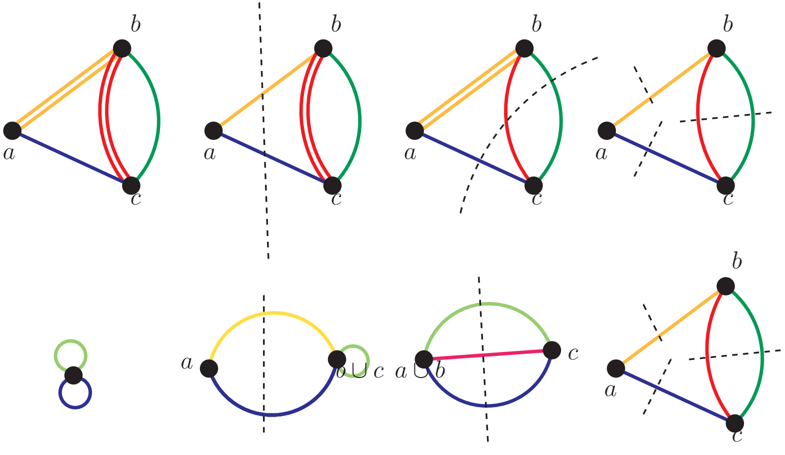


Corners A, B, C, D not part of OS.

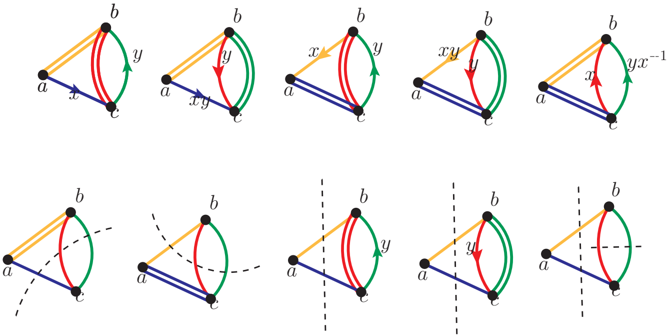
The Hasse diagram of a partition of vertices relates to an ordering



of the edges of T:

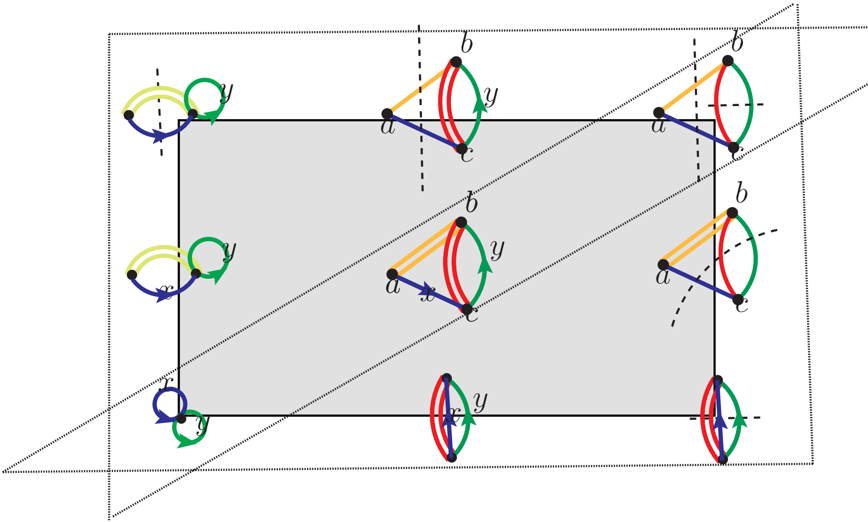


In fact, it is worth to consider all five spanning trees of the Dunce's cap given with spanning trees and markings:

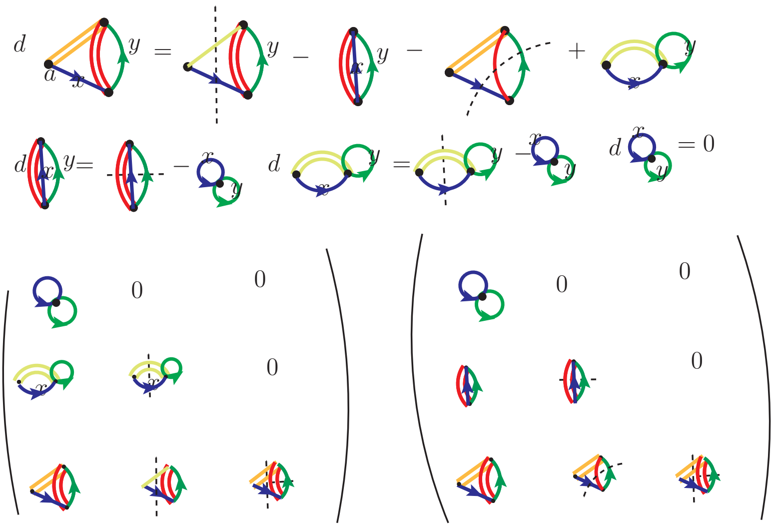


With five spanning trees each having two edges we get ten edge-ordered spanning trees. Three of them give rise to the leftmost Cutkosky cut in the lower row, and three of them to the second graph from left. The next two graphs in the bottow row refer to the same Cutkosky cut, but this time the internal edges connecting the vertices b, c on one side of the partition form a loop, which has two possible spanning trees, and both graphs can be generated by two of the ten edge-ordered spanning trees, which completes the tally.

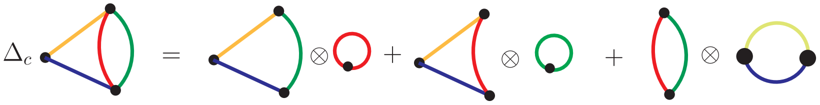
The boundary operator d of the cubical cell complex for the pair of the Dunce's cap with say the red-yellow spanning tree delivers the entries of a cube



This cube then delivers the Hodge matrices from its components:



This iteration of subgraphs is governed by the co-action $\Delta_c : H_\perp \to H_\perp \otimes H$ of the core coproduct Δ_c on the Hopf algebra of tadpole free Feynman graphs $H_\perp = H/H_\Omega$:



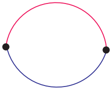
The terms on the right correspond to three flags of sub-/co-graphs, corresponding to three possible ways of computing the amplitude as an iterated integral over 1-loop subgraphs. That the results agree along principal sheets needs OPE and locality to work.

$$\Phi_R^{\mathrm{MV}}(G) \sim \left\{ \dot{\bigcup}_{\mathrm{flags}\, F} \Phi_R^{\mathrm{MV}}(f_1) \circ \cdots \circ \Phi_R^{\mathrm{MV}}(f_{|G|}) \right\},$$

where $F = \{f_1, \ldots, f_{|G|}\} \to \int d^4 l_{|G|} \Phi_R^{\mathrm{mv}}(f'_{|G|}) \ldots \int d^4 l_1 \Phi_R^{\mathrm{mv}}(f_1)$.
We need $\Phi_R(H_\Omega) = 0$

the bubble b_2

We start with the 2-edge banana, a bubble on two edges with two different internal masses m_b, m_r, indicated by two different colours:



We define the Kàllen function

$$\lambda(a,b,c) := a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

and find by explicit integration

$$\Phi_R(b_2)(s, s_0; m_r^2, m_b^2) =$$

$$= \left(\underbrace{\frac{\sqrt{\lambda(s, m_r^2, m_b^2)}}{2s} \ln \frac{m_r^2 + m_b^2 - s - \sqrt{\lambda(s, m_r^2, m_b^2)}}{m_r^2 + m_b^2 - s + \sqrt{\lambda(s, m_r^2, m_b^2)}} - \frac{m_r^2 - m_b^2}{2s} \ln \frac{m_r^2}{m_b^2}}_{B_2(s)} \right.$$

$$\left. - \underbrace{\{s \to s_0\}}_{B_2(s_0)} \right).$$

It is particularly interesting to compute the variation using Cutkosky's theorem

$$\mathrm{Var}(\Phi_R(b_2)) = 4\pi \int_0^\infty \sqrt{t}dt \int_{-\infty}^{\infty} dk_0 \delta_+(k_0^2 - t - m_r^2)\delta_+((k_0 - q_0)^2 - t - m_b^2).$$

The integral gives

$$\mathrm{Var}(\Phi_R(b_2))(s, m_r^2, m_b^2) = \overbrace{\left(\frac{\sqrt{\lambda(s, m_r^2, m_b^2)}}{2s}\right)}^{=:V_{rb}(s;m_r^2,m_b^2)} \times \Theta(s - (m_r + m_b)^2).$$

Note $\lambda(s, m_r^2, m_b^2) = (s - (m_r + m_b)^2)(s - (m_r - m_b)^2)$. We regain $\Phi_R(b_2)$ from $\mathrm{Var}(\Phi_R(b_2))$ by a subtracted dispersion integral:

$$\Phi_R(b_2) = \frac{s - s_0}{\pi} \int_0^\infty \frac{\mathrm{Var}(\Phi_R(b_2)(x))}{(x - s)(x - s_0)} dx,$$

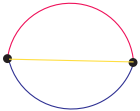
We define a multi-valued function

$$\Phi_R(b_2)^{\mathrm{mv}}(s, m_r^2, m_b^2) := \Phi_R(b_2)(s, m_r^2, m_b^2) + 2\pi\imath\mathbb{Z}V_{rb}(s).$$

Splitting:
$s < (m_r - m_b)^2, (m_r - m_b)^2 < s < (m_r + m_b)^2, (m_r + m_b)^2 \leq s.$

b_3

We now consider the 3-edge banana b_3 on three different masses.



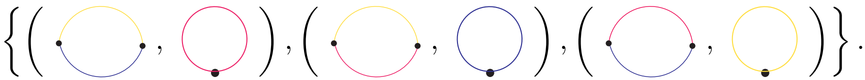
The resulting function $\Phi_R(b_3)$ has a structure similar to the dilogarithm function $\mathrm{Li}_2(z)$. As a multi-valued function, we can write the latter as

$$\mathrm{Li}_2^{\mathrm{mv}}(z) = \mathrm{Li}_2(z) + 2\pi\imath\mathbb{Z}\ln z + (2\pi\imath)^2\mathbb{Z}\times\mathbb{Z}.$$

We will find multi-valued functions

$$\begin{aligned} \mathrm{I}_k^{ij}(n_1,n_2)(s) &= \Phi_R(b_3)(s) + 2\pi\imath n_1 \int \frac{V_{ij}(k^2)(k^2;m_i^2,m_j^2)}{(k+q)^2-m_k^2} d^4k \quad (2)\\ &+(2\pi\imath)^2\frac{|m_k^2-s||m_i^2-m_j^2|}{2s}n_1n_2. \end{aligned}$$

We regard $\mathrm{I}_r^{by}(n_1,n_2)(s) \sim \mathrm{I}_b^{yr}(n_1,n_2)(s) \sim \mathrm{I}_y^{rb}(n_1,n_2)(s)$ as equivalent, with equivalence established by equality along the principal sheet.



Let us compute

$$\mathrm{Var}(\Phi_R(b_3)(s) = \int d^4k d^4l \delta_+(k^2 - m_b^2)\delta_+(l^2 - m_r^2)\delta_+((k - l + q)^2 - m_y^2).$$

Using Fubini, this can be written in three different ways in accordance with the flag structure:

$$\mathrm{Var}(\Phi_R(b_3)) = \int d^4k \mathrm{Var}(\Phi_R(b_2))(k^2, m_r^2, m_b^2)\delta_+((k + q)^2 - m_y^2),$$

or

$$\mathrm{Var}(\Phi_R(b_3)) = \int d^4k \mathrm{Var}(\Phi_R(b_2))(k^2, m_b^2, m_y^2)\delta_+((k + q)^2 - m_r^2),$$

or

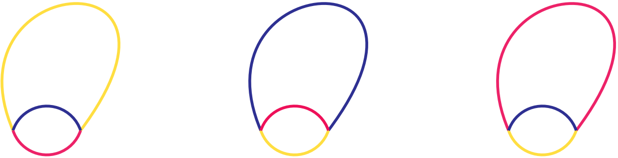
$$\mathrm{Var}(\Phi_R(b_3)) = \int d^4k \mathrm{Var}(\Phi_R(b_2))(k^2, m_y^2, m_r^2)\delta_+((k + q)^2 - m_b^2).$$

To study the sheet structure for b_3 we now define three different multi-valued functions as promised above

$$I_k^{ij} = I_k^{ji} = \int \frac{\Phi_R^{\mathrm{mv}}(b_2)(k^2, m_i^2, m_j^2)}{(k+q)^2 - m_k^2 + i\eta} d^4k,$$

with subtractions at $s = s_0$ understood as always such that the integrals exist.

For later use in the context of Outer Space we represent them as



I_y^{br} I_b^{yr} I_r^{by}

It is convenient to rewrite them as

$$I_k^{ij} = \int \frac{\Phi_R(b_2)(k^2, m_i^2, m_j^2)}{(k+q)^2 - m_k^2 + i\eta} d^4k + 2\pi\imath\mathbb{Z} \sum_{u=1}^{3} J_k^{ij;u}$$

using the splitting above in the sum.

Consider

$$\Im(J_k^{ij;3})(s) = \int d^4k \frac{\Theta(k^2-(m_i+m_j)^2)\sqrt{\lambda(k^2,m_i^2,m_j^2)}}{2k^2}\delta_+((k+q)^2)-m_k^2).$$

One finds

$$\Im(J_k^{ij;3})(s) = \int_0^{\frac{\lambda(s,m_k^2,(m_i+m_j)^2)}{4s}} \frac{\sqrt{\lambda(s+m_k^2-2\sqrt{s}\sqrt{t+m_k^2},m_i^2,m_j^2)}}{2(s+m_k^2-2\sqrt{s}\sqrt{t+m_k^2})\sqrt{t+m_k^2}}\sqrt{t}dt.$$

Note that the integrand vanishes at the upper boundary $\frac{\lambda(s,m_k^2,(m_i+m_j)^2)}{4s}$, and the integral has a pole at $s=0$, for $s=0$ the integral would not converge. The integrand is positive definite in the interior of the integration domain and free of singularities.

Most interesting is the computation of $\Im(J_k^{ij;1})(s)$. It gives

$$\Im(J_k^{ij;1})(s) = \int_{\frac{\lambda(s,m_k^2,(m_i-m_j)^2)}{4s}}^{\infty} \frac{\sqrt{\lambda(s+m_k^2-2\sqrt{s}\sqrt{t+m_k^2}, m_i^2, m_j^2)}}{2(s+m_k^2-2\sqrt{s}\sqrt{t+m_k^2})\sqrt{t+m_k^2}} \sqrt{t} dt.$$

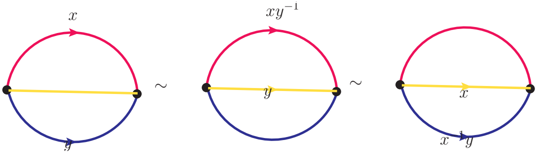
The integrand vanishes at the lower boundary $\frac{\lambda(s,m_k^2,(m_i-m_j)^2)}{4s}$, and the integral again has a pole at $s=0$. But now the integrand has a pole as $q_0^2+m_k^2-2q_0\sqrt{t+m_k^2}$ is only constrained to $\leq (m_i-m_j)^2$, and hence can vanish in the domain of integration. This gives us a new variation apparent in the integration of the loop in the co-graph

$$\mathrm{Var}(J_k^{ij;1})(s) = \int \sqrt{\lambda(k^2, m_i^2, m_j^2)} \delta(k^2) \delta_+((k+q)^2 - m_k^2) d^4k,$$

which evaluates to

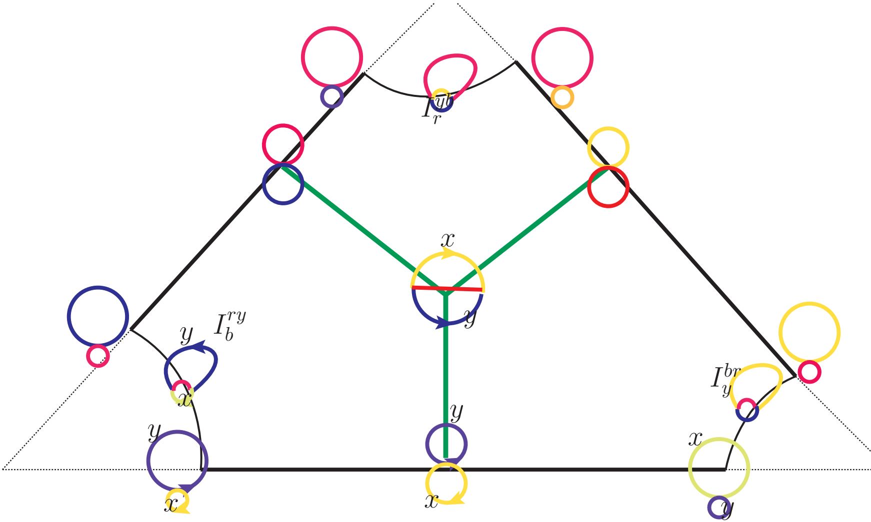
$$\mathrm{Var}(J_k^{ij;1})(s) = |m_i^2 - m_j^2| \frac{|s-m_k^2|}{2s} \Theta(s-m_k^2).$$

Consider the equivalence relation for b_3 in Outer Space.



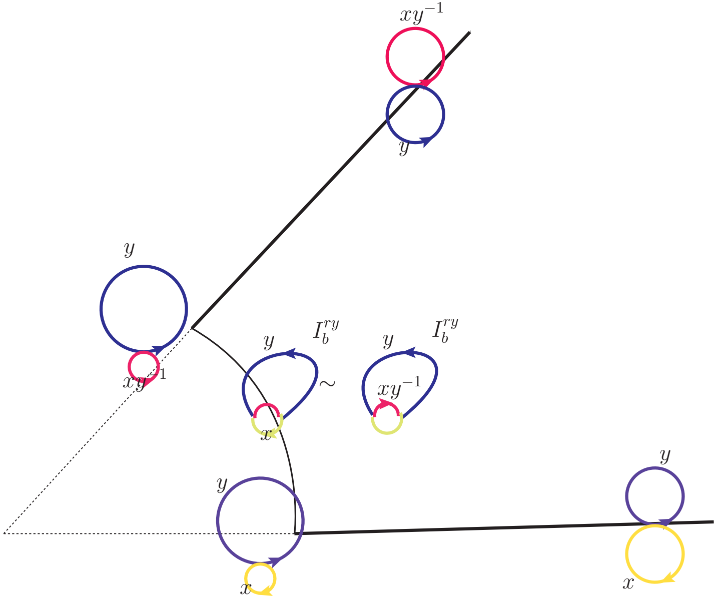
The three possible choices for a spanning tree of b_3 result in three different but equivalent markings of b_3 regarded as a marked metric graph in (coloured) Outer Space.
Each different choice corresponds to a different choice of basis for $H^1(b_3)$.
The choice of a spanning tree together with an ordering of the roses then determines uniquely a single element in the set of ordered flags of subgraphs, and hence determines one iterated Feynman integral describing the amplitude in question.
For their evaluation along principal sheets equality of these integrals follows by Fubini. This implies an equivalence relation for evaluation along the non-principal sheets.
Hope: On the level of amplitudes, a basis for the fundamental group of the graph, provided by a marking, translates to a basis for the fundamental group for the complement of the threshold divisors of the graph.

For b_3, we get two generators. A choice as which two edges form the subgraph b_2 then determines the iterated integral. The equivalence of markings in Outer Space becomes the corresponding equivalence of iterated integrals else.



Markings only partially given.

Let us have a still closer look at the corners:



The equivalence relation is an equivalence relation for the two marked metric graphs, which is indeed coming from an equivalence relation for the two choices of a spanning tree for the 2-edge subgraph on the red and yellow edges, while the corresponding analytic expression is for both choices I_b^{ry}.

Moving to a different corner by shrinking the size of the blue edge and increasing say the size of the red edge moves to a different corner while leaving the marking equal. This time we have an equivalence relation between the analytic expressions:

$$I_b^{ry} \sim I_r^{by}.$$

Moving along an arc uses equivalence based on homotopy of the graph, moving along an edge leaves the marking equal, but uses equivalence of analytic expressions $I^{\gamma}_{\Gamma/\gamma}$, here $I_b^{ry} \sim I_r^{by}$.

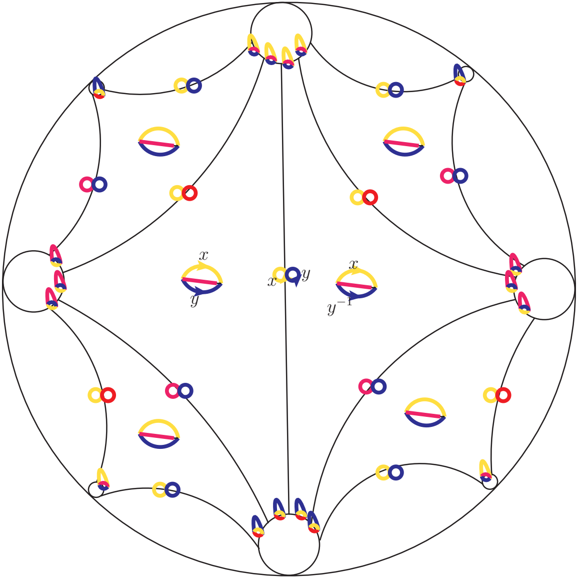
In this example the cograph was always a single-edge tadpole whose spanning tree is a single vertex and therefore the equivalence relation from the 1-petal rose R_1 to the co-graph is in fact the identity. In general, the decomposition of a graph into a subgraph γ and cograph Γ/γ corresponds to a factorization into equivalence classes for the subgraph and equivalence classes for the cograph familiar from Conant/Hatcher/Kassabov/Vogtmann.

$$\begin{pmatrix} 1 & 0 & 0 \\ \overbrace{\Phi_R^{\mathrm{mv}}(b_2)(sm_r^2, m_y^2)}^{\text{monodromy } (m_r \pm m_y)^2} & V_{ry}(s, m_r^2, m_y^2) & 0 \\ \overbrace{I_b^{ry}}^{\text{monodromy } (m_b \pm |m_r \pm m_y|)^2} & \overbrace{\Im \sum_{u=1}^{3} J_b^{ry;u}}^{\text{monodromy } u=1\text{: } m_b^2} & \frac{|m_r^2 - m_y^2||s - m_b^2|}{2s} \end{pmatrix}$$

The new monodromy at $s = m_b^2$ comes from the fact that V_{ry} has a pole at $k^2 = 0$, which generates (off the principal sheet) a pinching in $J_b^{ry,1}$.

The appearance of such mass independent poles off principal sheets a general phenomenon subject to linear reduction in the parametric representation (use $\psi_G = \phi_G / A \cdot M$).

The bordification of Outer Space as studied by Bux/Smillie/Vogtmann motivates to glue the cell studied above to a 'jewelled space':



Degrees

Momentum conservation for external edges at a graph G allows to connect them to a new vertex v_∞, resulting in complexes for graphs G_∞ or G^∞ with a distinguished base-point. Such complexes can be filtered using degrees:

$$||G|| := 2|G_\infty| - val(v_\infty) \equiv 2|G| + ext(G) - 2,$$

Under the coproduct the two variants behave similarly:

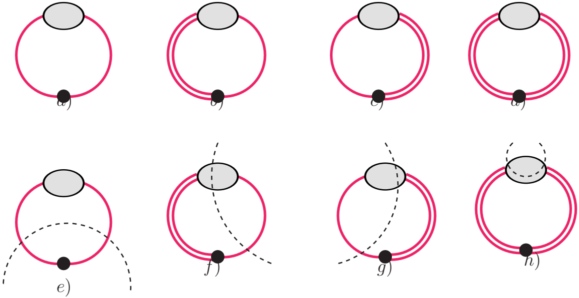
$$|||G||| := 2|G^\infty| - v_G \equiv 2|G| + v_G - 2.$$

$$||G|| = ||G'|| + ||G''|| - ||res(G')||,$$

$$|||G||| = |||G'||| + |||G''||| - |||res(G')|||.$$

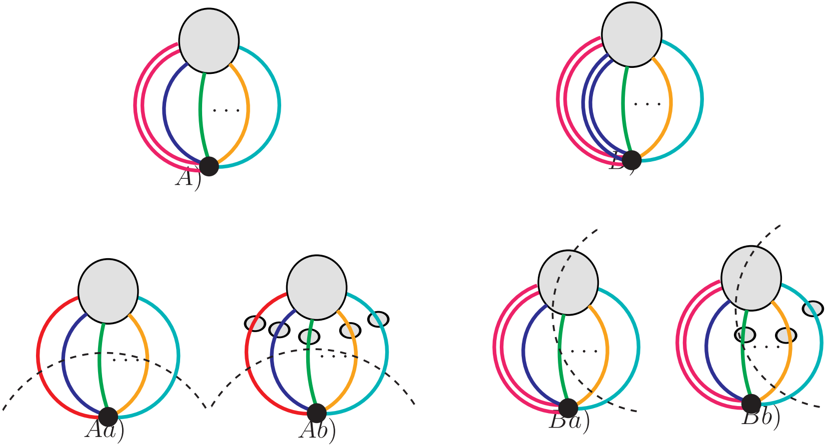
upon shrinking an edge though, $||G|| = ||G/e||$, whilst $|||G||| = |||G/e||| + 1$.

LSZ



The grey blob is any graph with two external edges (in red), connecting to the distinguished vertex v_∞ (the black dot). Spanning trees cover one or both of the two red edges. Removing an edge from the spanning tree results in a Cutkosky cut (lower row) which either puts both external edges on the mass-shell (leftmost graph), or at least one (the next two), or corresponds to a 2-partition of vertices such that both external edge couple to the same component (rightmost graph).
In the first case, we get zero as a renormalized self-energy vanishes on-shell, for the next two we get zero as the derivative of a renormalized graph also vanishes on-shell, and the rightmost vanishes trivially.

Thus, the vertex v_∞ must have valence greater than two, and therefore all vertices have valence ≥ 3. In conclusion we get the LSZ formalism:



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- Recursive/iterative structures similar for Outer Space and Amplitudes
- Task: repeat exercise including spin and other assignments from representation theory for edges and vertices. Compute cohomologies underlying CCC.

Parametric Wonderland

- $$\Phi(\Gamma) = \underbrace{\Phi(\Gamma/\gamma)\psi(\gamma)}_{k-1} + \underbrace{\Phi(\Gamma-\gamma)\psi(r_k) - M(\gamma)\psi(\Gamma/\gamma)\psi(\gamma)}_{k} - \underbrace{M(\gamma)\psi(\Gamma-\gamma)\psi(r_k)}_{k+1}$$

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- $$\phi^r_{x,y}(\Gamma) = \sum_{T_1 \cup T_2} (Q(T_1)\cdot Q(T_2))^r \prod_{e \notin T_1 \cup T_2} A_e,$$
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- $$\Phi(\Gamma-\gamma)E^\gamma_k - M(\gamma)\psi(\Gamma/\gamma)E^\gamma_{k-1} = \Phi^u(\Gamma-\gamma),$$
with $u = (\sum_{e\in E_\gamma} m_e)^2$.

From Discriminants to anomalous thresholds

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- iii) If for all $T \in T_s^\Gamma$ and for all their forests (Γ, F) we have $s_F > -\infty$, the Feynman integral $\Phi_R(\Gamma)(s)$ is real analytic as a function of s for $s < \min_F\{s_F\}$.

Example: The triangle

$$\Phi_\Delta = \overbrace{p_a^2 A_1 A_2 - (m_1^2 A_1 + m_2^2 A_1)(A_1 + A_2)}^{=\Phi_{\Gamma/e_3}} + A_3((p_b^2 - m_3^2 - m_1^2)A_1 + (p_c^2 - m_1^2 - m_3^2$$

so

$$\Phi_\Delta = \Phi_{\Delta/e_3} + A_3 \Phi^{m_3^2}_{\Delta - e_3} - A_3^2 m_3^2 \overbrace{\psi_{\Delta - e_1}}^{=1},$$

as announced ($A_3 = t_\gamma$):

$$X = \Phi_{\Delta/e_3},\ Y = \overbrace{(p_b^2 - m_3^2 - m_1^2)}^{=:l_1} A_1 + \overbrace{(p_c^2 - m_1^2 - m_3^2)}^{=:l_2} A_2,\ Z = m_3^2.$$

We have $Y_0 = m_2 l_1 + m_1 l_2$, and need $Y_0 > 0$ for a Landau singularity.

cont'd

Solving $\Phi(\Delta/e_3) = 0$ for a Landau singularity determines the familiar physical threshold in the $s = p_a^2$ channel, leading for the reduced graph to

$$p_Q : s_0 = (m_2 + m_3)^2, \; p_A : A_1 m_1 = A_2 m_2.$$

We let $D = Y^2 + 4XZ$ be the discriminant. For a Landau singularity we need

$$D = 0.$$

We have

$$\Phi_\Delta = -m_3^2 \left(A_3 - \frac{Y + \sqrt{D}}{2m_3^2} \right) \left(A_3 - \frac{Y - \sqrt{D}}{2m_3^2} \right),$$

where Y, D are functions of A_1, A_2 and $m_1^2, m_2^2, m_3^2, s, p_b^2, p_c^2$.

cont'd

We can write

$$0 = D = Y^2 + 4Z(sA_1A_2 - N),$$

with $N = (A_1 m_1^2 + A_2 m_2^2)(A_1 + A_2)$ s-independent. This gives

$$s(A_1, A_2) = \frac{4ZN - (A_1 l_1 + A_2 l_2)^2}{4ZA_1A_2} =: \frac{A_1}{A_2}\rho_1 + \rho_0 + \frac{A_2}{A_1}\rho_2.$$

Define two Kallen functions $\lambda_1 = \lambda(p_b^2, m_1^2, m_3^2)$ and $\lambda_2 = \lambda(p_c^2, m_2^2, m_3^2)$. Both are real and non-zero off their threshold or pseudo-threshold. Then, for

$$\lambda_1, \lambda_2 > 0,$$

we find the threshold s_1 at

$$s_1 = \frac{4m_3^2(\sqrt{\lambda_1}m_1^2 + \sqrt{\lambda_2}m_2^2)(\sqrt{\lambda_1} + \sqrt{\lambda_2}) - (\sqrt{\lambda_1}l_2 + \sqrt{\lambda_2}l_1)^2}{4m_3^2\sqrt{\lambda_1}\sqrt{\lambda_2}}.$$

cont'd

On the other hand for $r < 0$ and therefore the coefficients of ρ_1, ρ_2 above of different sign we find a minimum

$$s_1 = -\infty, \tag{3}$$

along either $A_1 = 0$ or $A_2 = 0$. Get dispersion from other channels, looking at other spanning trees, that is.

Things are not simpler than they can be, and not more difficult than they must be.