Kinetic and stochastic models of condensation

Stefan Grosskinsky

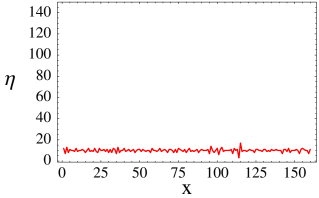
Warwick Mathematics Institute

July 12, 2019

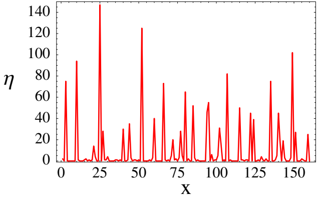
HIM, Bonn

Dynamics of condensation

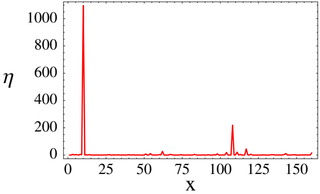
ZRP with $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/(b-2) = 0.5$, $\rho = 10$



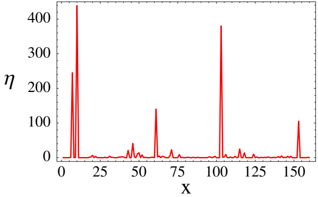
Nucleation → $O(1)$



$O(1)$ ↓ **Coarsening**



Saturation ← $O(L^2, L^3)$



hydrodynamics $O(L, L^2)$ [subcritical, Stamatakis (2015)]

stationary dynamics of condensate $O(L^{1+b}),\ O(N^{1+b})$

Previous rigorous results on condensation dynamics

Stationary dynamics for ZRP (metastability)

- L fixed, $N \to \infty$, $p(x,y)$ reversible [Beltrán, Landim (2010,11,12,15)]

$$Y^N\big(\eta(tN^{1+b})\big) \to Y_t \quad \text{RW on (subset of) } \Lambda \ , \quad \text{rates } \propto \mathrm{cap}_\Lambda(x,y)$$

  L fixed, $N \to \infty$, $p(x,y)$ asymmetric [Landim (2014), Seo (2018)]

- $L, N \to \infty$, $N/L \to \rho > \rho_c$, $p(x,y)$ symmetric on rescaled torus $\subset \mathbb{T}$

$$Y^L\big(\eta(tL^{1+b})\big) \to Y_t \quad \text{Lévy-type on } \mathbb{T}$$

[Armendáriz, G., Loulakis (2017)]
[Bovier, Neukirch (2014)]

Nucleation/Coarsening for ZRP

- L fixed, $N \to \infty$, $p(x,y)$ irreducible [Beltrán, Jara, Landim (2017)]

$$\boldsymbol{\eta}(tN^2)/N \to \mathbf{X}_t \quad \text{absorbed diffusion on } \Delta_L$$

Inclusion process

- L fixed, $N \to \infty$, $d = d_N \ll 1/\log N$, time scale t/d_N

  Coarsening for $p(x,y)$ symmetric, $Nd_N \to \infty$ [G., Redig, Vafayi (2013)]

  Stat. dynamics with multiple scales [Bianchi, Dommers, Giardiná (2017)]

Nucleation and coarsening/MF limits

Jatuviriyapornchai, G., Stoch. Proc. Appl. **129**(4), 1455-1475 (2019)

Jatuviriyapornchai, G., J. Phys. A: Math. Theor. **49**(18), 185005 (2016)

Godréche, Drouffe, J. Phys. A: Math. Theor. **50**(1), 015005 (2016)

Godréche, Luck, J. Phys. A: Math. Theor. **38**(33), 7215 (2005)

Y.-X. Chau, C. Connaughton, S. G., J. Stat. Mech., P11031 (2015)

J. Cao, P. Chleboun, S. G., J. Stat. Phys. **155**, 523543 (2014)

S. G., F. Redig, K. Vafayi, Electron. J. Probab. **18**, 123 (2013)

Mean-field equation

SPS with generator $$\mathcal{L}f(\boldsymbol{\eta})=\sum_{x,y\in\Lambda} p(x,y)u(\eta_x,\eta_y)\big(f(\boldsymbol{\eta}^{x,y})-f(\boldsymbol{\eta})\big)$$

Empirical measures $$F_k^L(\eta)=\frac{1}{L}\sum_{x\in\Lambda}\delta_{\eta_x,k}\in[0,1]$$

Assume

- **complete graph** $p(x,y)=1/(L-1)$
- **jump rates** $u(k,l)\leq C_1 k(C_2+l)$
- **initial conditions** $\eta(0)$ such that $F_k^L(\eta(0))\to f(0)$ on $\mathbb{N}_0$

  $$m_0(0)=1\ ,\quad m_1(0)=\sum_k k f_k(0)=\rho<\infty\ ,\quad m_2(0)<\infty$$

  and $\alpha_1,\ \alpha_2>0$ such that for all $L\geq 1$

  $$\eta(0)\in\Omega_\alpha:=\Big\{\eta:\frac{1}{L}\sum_{x\in\Lambda}\eta_x<\alpha_1,\ \frac{1}{L}\sum_{x\in\Lambda}\eta_x^2<\alpha_2\Big\}$$

  $\to$ for example $\eta_x(0)\sim f(0)$ i.i.d. bounded

Mean-field equation

Theorem – LLN for empirical process

Under above assumptions, for all $k \in \mathbb{N}_0$ the empirical processes $\big(F_k^L(\eta(t)) : t \geq 0\big)$ converge weakly on path space to $\big(f_k(t) : t \geq 0\big)$ as $L \to \infty$, which are given as the unique solution of the **mean-field (rate) equation**

$$\frac{d}{dt} f_k(t) = \sum_{l \geq 0} \Big(u(k+1,l) f_l(t) f_{k+1}(t) + u(l,k-1) f_l(t) f_{k-1}(t) \Big) - \sum_{l \geq 0} \big(u(k,l) + u(l,k) \big) f_l(t) f_k(t) \quad \text{for all } k \geq 0 \ , \qquad \text{(MFE)}$$

with initial condition $f(0)$ given above.

Mean-field equation

Theorem – LLN for empirical process

Under above assumptions, for all $k \in \mathbb{N}_0$ the empirical processes $\big(F_k^L(\eta(t)) : t \geq 0\big)$ converge weakly on path space to $\big(f_k(t) : t \geq 0\big)$ as $L \to \infty$, which are given as the unique solution of the **mean-field (rate) equation**

$$\frac{d}{dt} f_k(t) = \sum_{l \geq 0} \Big(u(k+1, l) f_l(t) f_{k+1}(t) + u(l, k-1) f_l(t) f_{k-1}(t) \Big) - \sum_{l \geq 0} \big(u(k,l) + u(l,k) \big) f_l(t) f_k(t) \quad \text{for all } k \geq 0 \,, \qquad \text{(MFE)}$$

with initial condition $f(0)$ given above.

- This implies in particular uniqueness of the solution to (MFE) for given $f(0)$,
- as well as convergence of expectations,

$$f_k^L(t) := \mathbb{E}^L\big[F_k^L(\eta(t))\big] = \frac{1}{L} \sum_{x \in \Lambda} \mathbb{P}^L\big[\eta_x(t) = k\big] \to f_k(t) \ .$$

Propagation of chaos

Assume in addition symmetry of initial conditions, i.e.

the law of $\{\eta_x(0) : x \in \Lambda\}$ is permutation invariant for each $L \geq 1$.

Corollary – Propagation of chaos (see e.g. [e.g. dai Pra (2017)])

Under the conditions of the Theorem and above, for any finite-dimensional marginal with distinct $x_1, \ldots, x_m \in \Lambda$, $m \geq 1$, we have for any $T > 0$

$$\big(\eta_{x_i}(t) : t \in [0, T]\big) \text{ converge to independent birth-death chains} \ ,$$

with (non-linear) master equation (MFE) and generator

$$\mathcal{L}_{f(t)} h(k) = \alpha_k(t)[h(k+1) - h(k)] + \beta_k(t)[h(k-1) - h(k)] \ ,$$

with rates $\alpha_k(t) = \sum_{l \geq 0} u(l, k) f_l(t)$ and $\beta_k(t) = \sum_{l \geq 0} u(k, l) f_l(t)$.

[Gärtner (1988) WASEP; Rezakhanlou (1994) SSEP and ZRP, (1996) multi-type model ...]

Proof of main result

1. existence of limits $t \mapsto f(t)$ via tightness
2. limits are solutions of (MFE)
3. uniqueness of solutions of (MFE)

Proof of main result

1. existence of limits $t \mapsto f(t)$ via tightness
2. limits are solutions of (MFE)
3. uniqueness of solutions of (MFE)

Moments. $m_n^L(t) := \mathbb{E}^L\Big[\sum_{k\geq 0} k^n F_k^L(\eta(t))\Big] = \sum_{k\geq 0} k^n f_k^L(t)$

$$m_0^L(t) \equiv 1 \quad \text{and} \quad m_1^L(t) \equiv m_1^L(0) \stackrel{L\to\infty}{\longrightarrow} \rho \ .$$

Lemma. $C > 0$ such that $m_2^L(t) \leq (\alpha_2 + Ct)e^{Ct}$ for all $t \geq 0$, $L \geq 1$,

$$\text{using} \qquad \frac{d}{dt}\mathbb{E}^L\big[F_k^L(\eta(t))\big] = \mathbb{E}^L\big[\mathcal{L}F_k^L(\eta(t))\big] \quad \text{and Gronwall .}$$

Proof of main result

1. Tightness. For each bounded $h : \mathbb{N}_0 \to \mathbb{R}$ the law of

$$t \mapsto H(\eta(t)) := \sum_{k\geq 0} h_k F_k^L(\eta(t))$$

on path space $D_{[0,\infty)}(\mathbb{R})$ is tight as $L \to \infty$.

Using a version of **Aldous' criterion**, $\big|\sum_k h_k F_k^L(\eta)\big| \leq \|h\|_\infty$ and Markov's inequality we need to establish

$$\limsup_{L\to\infty} \sup_{t<\delta} \sup_{\zeta\in\Omega_\alpha} \mathbb{E}_\zeta^L\big[|H(\eta(t)) - H(\zeta)|\big] \to 0 \quad \text{as } \delta \to 0^+ \ .$$

Proof of main result

1. Tightness. For each bounded $h : \mathbb{N}_0 \to \mathbb{R}$ the law of

$$t \mapsto H(\eta(t)) := \sum_{k\geq 0} h_k F_k^L(\eta(t))$$

on path space $D_{[0,\infty)}(\mathbb{R})$ is tight as $L \to \infty$.

Using a version of **Aldous' criterion**, $\big|\sum_k h_k F_k^L(\eta)\big| \leq \|h\|_\infty$ and Markov's inequality we need to establish

$$\limsup_{L\to\infty} \sup_{t<\delta} \sup_{\zeta\in\Omega_\alpha} \mathbb{E}_\zeta^L\big[|H(\eta(t)) - H(\zeta)|\big] \to 0 \quad \text{as } \delta \to 0^+ \ .$$

Itô's formula $\quad M_h(t) := H(\eta(t)) - H(\eta(0)) - \int_0^t \mathcal{L}H(\eta(s))ds$

is a local m'gale with pred. QV $\quad \langle M_h\rangle(t) = \int_0^t [\mathcal{L}H^2 - 2H\mathcal{L}H](\eta(s))ds\ .$

$$\mathcal{L}H(\eta) = \sum_{k\geq 0} h_k \Big[F_{k-1}^L(\eta) \sum_{l\geq 1} u(l,k-1) F_l^L(\eta) + F_{k+1}^L(\eta) \sum_{l\geq 0} u(k+1,l) F_l^L(\eta)$$

$$- F_k^L(\eta) \sum_{l\geq 0} \big(u(k,l)+u(l,k)\big) F_l^L(\eta) \Big] (1+1/L) + \Delta_L(\eta)$$

Proof of main result

$$\mathbb{E}^L\left[\left|H(\eta(t)) - H(\eta(0))\right|\right] \leq \underbrace{\int_0^t \mathbb{E}^L|\mathcal{L}H(\eta(s))|ds}_{(1)} + \underbrace{\mathbb{E}^L\left[[M_h](t)\right]}_{(2)}.$$

with estimates

$$(1) \leq t\|h\|_\infty \Big(4C_1\alpha_1\left(\alpha_1 + C_2\right) + \frac{C}{L}(1+t)e^{Ct}\Big)$$

$$(2) \leq t\|h\|_\infty^2 \frac{1}{L}\Big(4C_1\alpha_1\left(\alpha_1 + C_2\right) + \frac{C}{L}(1+t)e^{Ct}\Big)$$

Both vanish as $t \leq \delta \to 0$ uniformly in L which implies **tightness**.

Proof of main result

$$\mathbb{E}^L\Big[\big|H(\eta(t)) - H(\eta(0))\big|\Big] \le \underbrace{\int_0^t \mathbb{E}^L|\mathcal{L}H(\eta(s))|ds}_{(1)} + \underbrace{\mathbb{E}^L\big[[M_h](t)\big]}_{(2)}.$$

with estimates

$$(1) \le t\|h\|_\infty\Big(4C_1\alpha_1\left(\alpha_1 + C_2\right) + \frac{C}{L}(1+t)e^{Ct}\Big)$$
$$(2) \le t\|h\|_\infty^2\frac{1}{L}\Big(4C_1\alpha_1\left(\alpha_1 + C_2\right) + \frac{C}{L}(1+t)e^{Ct}\Big)$$

Both vanish as $t \le \delta \to 0$ uniformly in L which implies **tightness**.

2. Estimate for (2) implies $\mathbb{E}^L[[M_h](t)] \to 0$ as $L \to \infty$ for all $t \ge 0$,
so the martingale vanishes and each limit **solves a weak version of (MFE)**

$$\sum_{k\ge 0} h_k\big(f_k(t) - f_k(0)\big) = \int_0^t \sum_{k\ge 0} h_k\big(\mathcal{L}^\dagger_{f(s)}f(s)\big)_k\, ds\ .$$

Proof of main result

3. Uniqueness of solutions of (MFE)

- **moments** $m_n(t) = \sum_{k\geq 0} k^n f_k(t)$

  $m_0(t) \equiv m_0(0)$ and $m_1(t) \equiv m_1(0) = \rho$ are conserved.

  Gronwall estimate $m_2^L(t) \leq (\alpha_2 + Ct)e^{Ct}$ for all $t \geq 0$.

- Consider $f(t), \hat{f}(t)$ with $f(0) = \hat{f}(0) \in \mathcal{P}(\mathbb{N}_0)$ and establish Gronwall for

$$\theta(t) := \sum_{k\geq 0}(k+1)\big|\Delta_k(t)\big| \quad \text{where} \quad \Delta_k(t) := f_k(t) - \hat{f}_k(t) .$$

[Esenturk (2017), Schlichting (2018)], following classical proof [Ball, Penrose (1986)]

Proof of main result

3. Uniqueness of solutions of (MFE)

- **moments** $m_n(t) = \sum_{k\geq 0} k^n f_k(t)$

$$m_0(t) \equiv m_0(0) \quad \text{and} \quad m_1(t) \equiv m_1(0) = \rho \quad \text{are conserved.}$$

Gronwall estimate $m_2^L(t) \leq (\alpha_2 + Ct)e^{Ct}$ for all $t \geq 0$.

- Consider $f(t), \hat{f}(t)$ with $f(0) = \hat{f}(0) \in \mathcal{P}(\mathbb{N}_0)$ and establish Gronwall for

$$\theta(t) := \sum_{k\geq 0}(k+1)\big|\Delta_k(t)\big| \quad \text{where} \quad \Delta_k(t) := f_k(t) - \hat{f}_k(t) \ .$$

[Esenturk (2017), Schlichting (2018)], following classical proof [Ball, Penrose (1986)]

Extensions. multiple particle jumps and quantitative bounds

systematic error $$\Big|\mathbb{E}^L\Big[H(\eta(t)) - \sum_{k\geq 0} h_k f_k(t)\Big]\Big| \leq Ce^{Ct}\frac{\|h\|_\infty}{L}$$

random error $$\mathbb{E}^L\Big[\big|H(\eta(t)) - \mathbb{E}^L[H(\eta(t))]\big|^2\Big]^{1/2} \leq Ce^{Ct}\frac{\|h\|_\infty}{\sqrt{L}}$$

Scaling analysis for ZRP

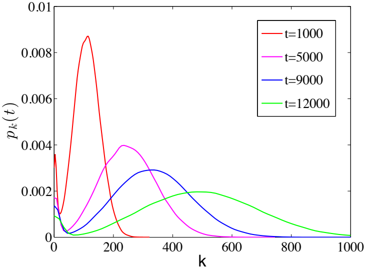
Scaling ansatz for phase separated solution with $m_1(t) = \rho > \rho_c$

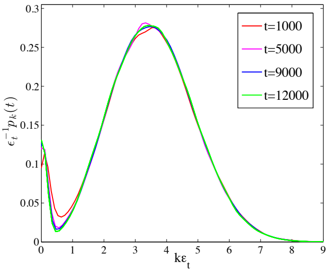
$$f_k(t) = f_k^{\mathrm{bulk}}(t) + \epsilon_t^2 h(k\epsilon_t) \quad \text{as } t \to \infty$$

with scale $\epsilon_t \to 0$ and scaling function $h(u)$, $u > 0$, and $h(u) \to 0$ as $u \to \infty$

We have $f^{\mathrm{bulk}}(t) \to f^{\rho_c}$ and $\displaystyle\sum_{k>0} k\epsilon_t^2 h(k\epsilon_t) \to \int_{u>0} uh(u)\,du = \rho - \rho_c$.

ZRP with rates $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/2$, $\rho = 10$

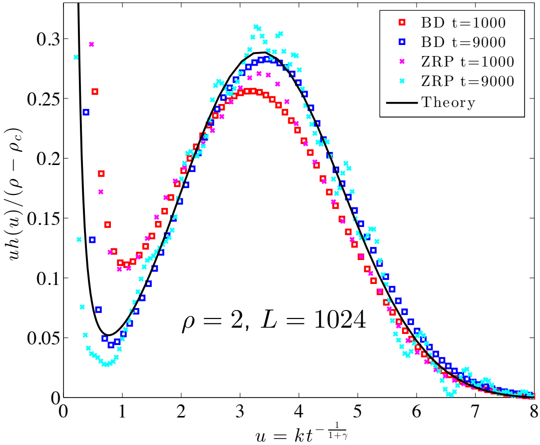




Scaling analysis for ZRP

$$\epsilon_t = t^{-1/2}\ , \quad h''(u) + \Big(\frac{u}{2} - A + \frac{b}{u}\Big)h'(u) + \Big(1 - \frac{b}{u^2}\Big)h(u) = 0$$

[Godréche (2003); J., G. (2016); Godréche, Drouffe (2016)]



Coarsening scaling law for ZRP

ZRP with rates $g(k) = 1 + b/k$ with $\rho_c = 1/(b-2)$.

Provided we can rigorously establish the scaling solution $f(t)$ for $m(0) = \rho > \rho_c$, we would have for the empirical measures with $\epsilon_t = t^{-1/2}$

$$\lim_{t\to\infty} \lim_{L\to\infty} \frac{1}{\epsilon_t^2} F^L_{[u/\epsilon_t]}(\boldsymbol{\eta}(t)) = h(u) \ ,$$

where h solves $h''(u) + \Big(\frac{u}{2} - A + \frac{b}{u}\Big) h'(u) + \Big(1 - \frac{b}{u^2}\Big) h(u) = 0$.

Coarsening scaling law for ZRP

ZRP with rates $g(k) = 1 + b/k$ with $\rho_c = 1/(b-2)$.

Provided we can rigorously establish the scaling solution $f(t)$ for $m(0) = \rho > \rho_c$, we would have for the empirical measures with $\epsilon_t = t^{-1/2}$

$$\lim_{t\to\infty}\lim_{L\to\infty}\frac{1}{\epsilon_t^2}F^L_{[u/\epsilon_t]}(\boldsymbol{\eta}(t)) = h(u) \ ,$$

where h solves $h''(u) + \Big(\frac{u}{2} - A + \frac{b}{u}\Big)h'(u) + \Big(1 - \frac{b}{u^2}\Big)h(u) = 0$.

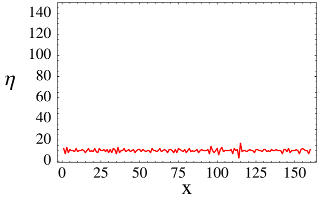
For size-biased samples $\bar{\eta}_x$ of occupation numbers this means

$$\lim_{t\to\infty}\lim_{L\to\infty}\epsilon_t\bar{\eta}_x(t) = \begin{cases} 0 \ , & \text{with prob. } \rho_c/\rho \\ U \ , & \text{with prob. } 1-\rho_c/\rho \end{cases} ,$$

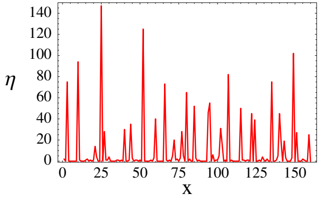
where U has density $uh(u)/(\rho - \rho_c)$.

Dynamics of condensation

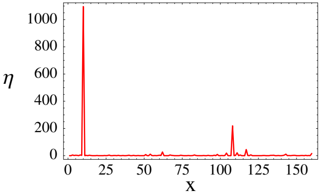
ZRP with $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/(b-2) = 0.5$, $\rho = 10$



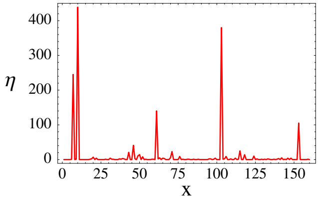
Nucleation → $O(1)$



$O(1)$ ↓ **Coarsening**



Saturation ← $O(L^2, L^3)$



hydrodynamics $O(L, L^2)$ [subcritical, Stamatakis (2015)]

stationary dynamics of condensate $O(L^{1+b}),\ O(N^{1+b})$

Stationary dynamics/metastability

Seo, CMP **366**(2), 781839 (2019)

Landim CMP **330**(1), 132 (2014)

Armendáriz, G., Loulakis. PTRF **169**(12), 105-175 (2017)

Bovier, Neukirch (2014)

Beltrán Landim PTRF **152**(3-4), 781807 (2012)

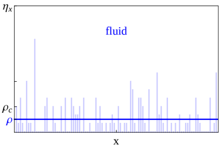
Canonical measures and condensation

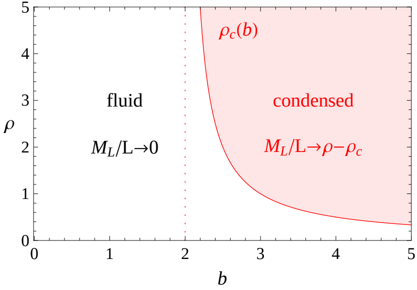
fixed number of particles N: $\pi_{L,N}[\,\cdot\,] = \frac{\nu_\phi[\,\cdot\cap\sum_x \eta_x = N]}{\nu_\phi[\sum_x \eta_x = N]}$

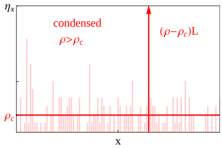
Equivalence of ensembles

In the thermodynamic limit $L, N \to \infty\ ,\quad N/L \to \rho$

$$\pi_{L,N} \to \nu_\phi \quad \text{where} \quad \begin{cases} R(\phi) = \rho\ , & \rho \le \rho_c \\ \phi = \phi_c\ , & \rho \ge \rho_c \end{cases} .$$







[Jeon, March, Pittel '00; G., Schütz, Spohn '03; Armendáriz, Loulakis '09 with G. '13; Chleboun, G. '14]

Metastability

Stationary dynamics of the condensate

$\Lambda = \mathbb{T}_L$, $g(k) = \mathbb{1}_{k>0}(1 + \frac{b}{k})$, $\quad b > 21$ (!) , $\quad p(x,y)$ NN symmetric

$M_L(\boldsymbol{\eta}) = \max_{x\in\Lambda} \eta_x$, $\quad \psi_L(\boldsymbol{\eta}) = \min\big\{x \in \Lambda : \eta_x = M_L(\boldsymbol{\eta})\big\}$

Metastability

Stationary dynamics of the condensate

$\Lambda = \mathbb{T}_L$, $g(k) = \mathbb{1}_{k>0}(1+\frac{b}{k})$, $b > 21$ (!) , $p(x,y)$ NN symmetric

$M_L(\boldsymbol{\eta}) = \max_{x\in\Lambda} \eta_x$, $\psi_L(\boldsymbol{\eta}) = \min\big\{x \in \Lambda : \eta_x = M_L(\boldsymbol{\eta})\big\}$

Theorem

Let $\boldsymbol{\eta}_0 \sim \pi_{L,N}$ and consider the thermodynamic limit $L, N \to \infty$ with $L/N \to \rho > \rho_c$. On the scale $\theta_L = L^{1+b}$, $\big(\frac{1}{L}\psi_L(\boldsymbol{\eta}_{\theta_L t}) : t \geq 0\big)$ converges weakly on path space $D\big([0,\infty), \mathbb{T}\big)$ to a Markov process $(Y_t : t \geq 0)$ with stationary, independent increments and generator

$$\mathcal{L}_{\mathbb{T}} f(u) = \int_{\mathbb{T}\setminus\{0\}} \big(f(u+v) - f(u)\big) \frac{C_{b,\rho}}{|v|(1-|v|)}\, dy$$

for all $f \in C^1(\mathbb{T})$.

$C_{b,\rho} = \big(\frac{b-1}{b}\big)(\rho-\rho_c)^b \Big(\Gamma(1+b)\int_0^{\rho-\rho_c} z^b(\rho-\rho_c-z)^b dz\Big)^{-1}$

Method of proof

Potential theory. [Bovier, Eckhoff, Gayrard, Klein (2001,2002) ... Bovier, den Hollander (2015)]

- **potential wells** $\mathcal{E}_x \subset \{\boldsymbol{\eta} : \psi_L(\boldsymbol{\eta}) = x\}$, $\pi_{L,N}(\cup_{x\in\Lambda}\mathcal{E}_x) \to 1$
  time spent outside wells can be ignored
- **effective rates** $R_L(x,y) = \mathbb{E}_{\pi_{L,N}|\mathcal{E}^x} \sum_{\boldsymbol{\zeta}} r(.,\boldsymbol{\zeta})\, \mathbb{P}_{\boldsymbol{\zeta}}(\boldsymbol{\eta}_\tau \in \mathcal{E}_y)$
  sharp bounds via capacities $\simeq C_{b,\rho}\mathrm{cap}_\Lambda(x,y)L/\theta_L$
  $\to$ bounds match only after regularization!

Method of proof

Potential theory. [Bovier, Eckhoff, Gayrard, Klein (2001,2002) . . . Bovier, den Hollander (2015)]

- **potential wells** $\mathcal{E}_x \subset \{\boldsymbol{\eta} : \psi_L(\boldsymbol{\eta}) = x\}$, $\pi_{L,N}(\cup_{x\in\Lambda}\mathcal{E}_x) \to 1$
  time spent outside wells can be ignored
- **effective rates** $R_L(x,y) = \mathbb{E}_{\pi_{L,N}|\mathcal{E}^x} \sum_{\boldsymbol{\zeta}} r(.,\boldsymbol{\zeta})\, \mathbb{P}_{\boldsymbol{\zeta}}(\boldsymbol{\eta}_\tau \in \mathcal{E}_y)$
  sharp bounds via capacities $\simeq C_{b,\rho}\mathrm{cap}_\Lambda(x,y)L/\theta_L$
  $\to$ bounds match only after regularization!

Martingale approach. [Landim, Beltran (2011,2012) . . .]

- **tightness** of $Y_t^L := \frac{1}{L}\psi_L(\boldsymbol{\eta}_{\theta_L t})$ as $L \to \infty$
  involves uniform upper bounds on rates (coupling)
- **martingale problem** for all $f \in C^1(\mathbb{T})$

$$f(Y_t^L) - f(Y_0^L) - \int_0^t \mathcal{L}f(Y_s^L)\,ds \quad \text{is a martingale}$$

- **equilibration** replace $\psi_L(\boldsymbol{\eta}_{\theta_L t})$ with a process on Λ with rates R_L
  show that $t_{\mathrm{rel}} \le CL^4$ and $t_{\mathrm{mix}}(\epsilon) \le CL^5 \log\frac{1}{\epsilon} \ll \theta_L$ on the well

Method of proof

Potential theory. [Bovier, Eckhoff, Gayrard, Klein (2001,2002) ... Bovier, den Hollander (2015)]

- **potential wells** $\mathcal{E}_x \subset \{\boldsymbol{\eta} : \psi_L(\boldsymbol{\eta}) = x\}$, $\pi_{L,N}(\cup_{x\in\Lambda}\mathcal{E}_x) \to 1$
  time spent outside wells can be ignored
- **effective rates** $R_L(x,y) = \mathbb{E}_{\pi_{L,N}|\mathcal{E}^x} \sum_{\boldsymbol{\zeta}} r(.,\boldsymbol{\zeta})\, \mathbb{P}_{\boldsymbol{\zeta}}(\boldsymbol{\eta}_\tau \in \mathcal{E}_y)$
  sharp bounds via capacities $\simeq C_{b,\rho}\mathrm{cap}_\Lambda(x,y)L/\theta_L$
  $\to$ bounds match only after regularization!

Martingale approach. [Landim, Beltran (2011,2012) ...]

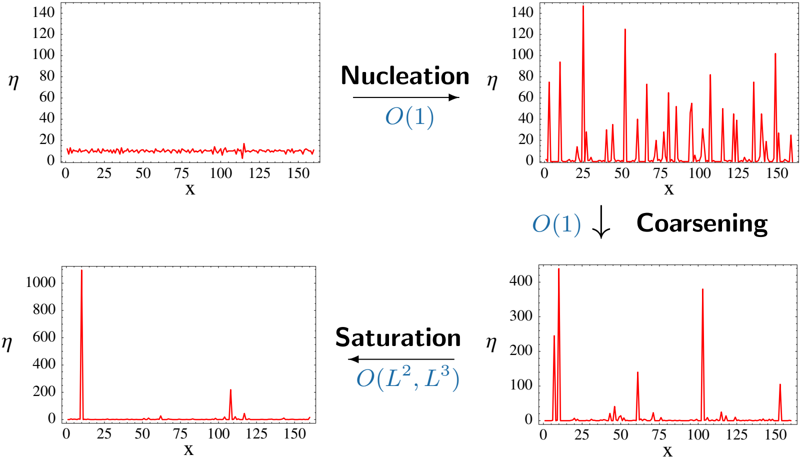
- **tightness** of $Y_t^L := \frac{1}{L}\psi_L(\boldsymbol{\eta}_{\theta_L t})$ as $L \to \infty$
  involves uniform upper bounds on rates (coupling)
- **martingale problem** for all $f \in C^1(\mathbb{T})$

$$f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}_{\mathbb{T}} f(Y_s)\, ds \quad \text{is a martingale}$$

- **equilibration** replace $\psi_L(\boldsymbol{\eta}_{\theta_L t})$ with a process on Λ with rates R_L
  show that $t_{\mathrm{rel}} \le CL^4$ and $t_{\mathrm{mix}}(\epsilon) \le CL^5 \log\frac{1}{\epsilon} \ll \theta_L$ on the well

Dynamics of condensation

ZRP with $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/(b-2) = 0.5$, $\rho = 10$



hydrodynamics $O(L, L^2)$ [subcritical, Stamatakis (2015)]

stationary dynamics of condensate $O(L^{1+b}),\ O(N^{1+b})$