$HC^{\cdot}(\mathcal{C})$-m

Singular Support

Slogan "Sing supp is support with respect to the action of Hochschild cochains"

Definition $\mathcal{C}$ dg-category

$$HC^{\cdot}(\mathcal{C}) = \operatorname{End}_{\operatorname{End}(\mathcal{C})}(\mathrm{id}_{\mathcal{C}})$$

$\mathbb{E}_2$-algebra

$HC(\mathcal{C})$-mod acts on $\mathcal{C}$

Definition

An $\mathbb{E}_2$-algebra A acts on $\mathcal{C}$ if there is a homomorphism

$$A \to HC^{\cdot}(\mathcal{C})$$

Assume $H(A)$ concentrated in Even degrees

$Y \subseteq \operatorname{Spec} H(A)$ closed

Definition

Category at Y

is

to

d cochains"

cls on $\mathcal{C}$

phism

)

Even degrees

Definition

Category of objects in $\mathcal{C}$ supported at Y with respect to an A-action

$$\leftrightarrow \quad \mathcal{C} \otimes_{A\text{-mod}} A\text{-mod}_Y$$

$A\text{-mod}_Y$: A-modules whose support meets Y

$$H(\Lambda) \longrightarrow HH(\mathcal{C}) \longrightarrow H^{\cdot}(\mathrm{End}_{\mathcal{C}}(c))$$

Support of c is the support of

$H^{\cdot}(\mathrm{End}_{\mathcal{C}}(c))$ wrt this module structure

Physics Interpretation

$\Sigma \times \mathbb{R}$

$\times \mathbb{R}_{\geq 0}$

4d topological field theory

A = local operators — $\mathbb{E}_2$ from directions perpendicular to Σ

$\mathcal{C}$ = category of boundary conditions on surface Σ

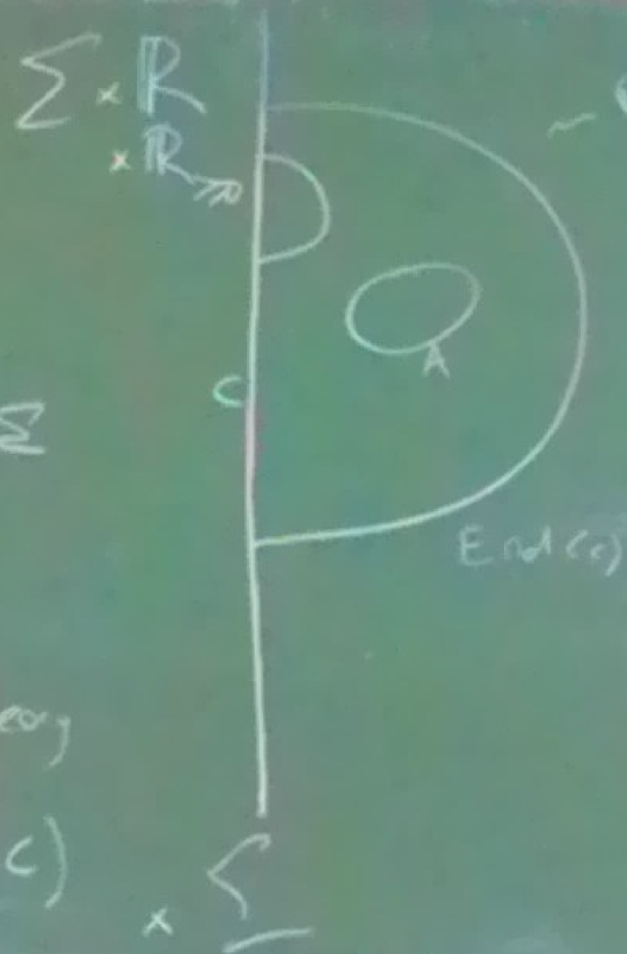
Choose a BC $c \in \mathcal{C}$

Model local observables in the theory coupled to c along Σ by $\mathrm{End}(c)$

...d vectors
...endicular to Σ

...n the theory

...y $End(c)$



picture of action
$A \otimes End(c) \to End(c)$

Interpret support
as C is supported
at point $v \in \operatorname{Spec} H(A)$
if the localization of
coupled local operators $End(c)$
at v is non-zero

Interpretation of Spec $H^\bullet(A)$

Def: A state is a functional $\mathrm{Obs} \xrightarrow{\phi} \mathbb{C}$ from local observables to numbers

A state ϕ is a vacuum if it is translation invt & it satisfies cluster decomposition

i.e.

If you have t
local observables

$\mathcal{O}_1 * \tau_x \mathcal{O}$

Operator p

If you have two

local observables $\mathcal{O}_1$ & $\mathcal{O}_2$

$$\underbrace{\mathcal{O}_1 * \tau_x \mathcal{O}_2}_{\text{Operator product}}(\phi) \longrightarrow \mathcal{O}_1(\phi)\,\mathcal{O}_2(\phi) \quad \text{as } x \to \infty$$

translate by $x \in \mathbb{R}^n$

...n map)

...nslation

<u>...decomposition</u>

In topological case this just says ϕ is a ring homomorphism

De...

$\phi) \mathcal{O}_2(\phi)$ as

$x \to \infty$

...al case

...s ϕ is

...orphism

<u>Definition</u>

A <u>vacuum</u> in a top^l theory is a point in $\mathrm{Spec}\, H^{\cdot}(A)$

Singular Support (Arinkin–Gaitsgory)

For the category $\mathcal{C} = \mathrm{IndCoh}(X)$ the action of HC ~~receives a map~~ from a more geometric object.

Definition The scheme of singularities of X is $(T^*[-1]X)^{cl}$

Classical part.

If X ...

(there's ...

$\mathcal{O}$ ...

So you ...

Support ...

$Y /$ ...

More gener...

...ts gory)

...oh (X)

~~...map~~

...

...ting

Classical pnt.

quasi-smooth

If X is ~~a~~ affine derived scheme

there's a map (not graded)

$$\mathcal{O}(\mathrm{Sing}\,X) \to HC(X)$$

So you can talk about singular support at closed subsets

$$Y \subseteq \mathrm{Sing}\,X$$

More generally can define this via descent

$X = \mathrm{Flat}$

... scheme

...not graded)

...$HC(X)$

...t singular

...osets

...this via

$$X = \mathrm{Flat}_G(\Sigma) \simeq \mathrm{Maps}(\Sigma_{dR}, BG)$$

$$\mathrm{Sing}(\mathrm{Flat}_G(\Sigma)) = \mathrm{Arth}_G(\Sigma)$$

has closed points

$$\left\{ (P, \nabla, \phi) \;\middle|\; \underset{\text{G-bundle}}{(P,\nabla) \text{ flat}},\ \underset{\text{flat section}}{\phi \in H^0_\nabla(\Sigma, \mathfrak{g}^*_P)} \right\}$$

Inside of $\mathrm{Arth}_G(\Sigma)$
there is the <u>global nilpotent cone</u>

$$\mathcal{N}_G = \{ (P, \nabla, \phi) \ \ ev_x \phi \text{ is nilpotent} \}$$

<u>Conjecture</u> (Arinkin-Gaitsgory)

$$D\text{-mod}(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

singular support on $\mathcal{N}_{G^\vee}$

Aim to show $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_{G^\vee}(\Sigma))$ will arise as cat of BCs in B-twist compatible with the Vacuum 0 in

$$\mathrm{Vac} = \mathrm{Spec}\, \mathcal{O}(\mathfrak{h}^*[2]/W)$$

$$= \mathfrak{h}^*[2]/W \ni \{0\}$$

$\Sigma))$

What of

$(\mathrm{Flat}_G(\Sigma))$

of BCs

tible with the

$^2]/W)$

$o\}$

What is the action of this algebra?

It arises from the action of Cat of <u>line operators</u> aka the action of Spectral Hecke category

Choose $x \in \Sigma$

Write $\mathbb{B}$ for $\mathbb{D} \cup_{\mathbb{D}^\times} \mathbb{D}$ "formal bubble"

$$
\begin{tikzcd}
 & \mathrm{Flat}_G(\mathbb{B}) & \\
 & \mathrm{Flat}_G(\Sigma \cup_{\mathbb{D}^\times_x} \mathbb{D}_x) \arrow[u, "q"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Flat}_G(\Sigma) & & \mathrm{Flat}_G(\Sigma)
\end{tikzcd}
$$

...al bubble"

$\mathcal{G} \in \mathrm{IndCoh}(\mathrm{Flat}_{\check{G}}(\mathbb{B}))$

acts on

$\mathcal{F} \in \mathrm{IndCoh}(\mathrm{Flat}_{\check{G}}(\Sigma))$

by integral transforms

$$p_{2*}\left(i^!(\mathcal{G}) \otimes p_1^!(\mathcal{F})\right)$$

Can check

$\mathrm{End}_{\mathrm{IndC}}$

$S.$

$\mathcal{B})) = \mathbb{1}$ Can check

$$\mathrm{End}_{\mathrm{IndCoh}(\mathrm{Fl}_G(\mathcal{B}))}(S_\cdot) \cong \mathcal{O}(\mathfrak{h}^*[z]/W)$$

$\Sigma)) \ni \mathcal{B}\, S_\cdot$

...sforms

$$\mathbb{1} \to \mathrm{End}(\mathcal{B})$$

$$= \mathrm{End}_{\mathbb{1}}(S_\cdot) \to \mathrm{End}_{\mathrm{End}\,\mathcal{B}}(\mathrm{id}) = HC(\mathcal{B})$$

$$\mathrm{End}_{\mathbb{1}}(S_\cdot) = \mathcal{O}(\mathfrak{h}^*[z]/W)$$

$))$

What do support conditions mean?

There's a map

$$\begin{tikzcd} \mathrm{Arith}_G(\Sigma) \arrow[d, "ev_x"] & \mathcal{N}_G \arrow[l] \arrow[dd] \\ \mathfrak{g}^*/G \arrow[d] & \\ \mathfrak{t}^*/W & \{0\} \arrow[l] \end{tikzcd}$$

induces a map

$\mathcal{O}(\mathfrak{t}^*/W) \to \mathcal{O}(\mathrm{Arith}_G(\Sigma))$

I claim support at $0 \in \mathfrak{t}^*/W$

$\iff$ support at $\mathcal{N}_G \subseteq \mathrm{Arith}_G(\Sigma)$

or

Theore

Theorem (E–Yoo)

The Arinkin–Gaitsgory cat is equivalent to BCs in B-twisted theory supported at 0 in $\mathfrak{h}^*[z]/W$

Rem

Removing the shift

One way of defining the twist of a field theory was to view the untwisted theory as a module over $\mathbb{C}[[t]]$

t deg 1 fermionic parameter

to invert t, then take $\mathbb{C}^\times$ invariants

Let's not take $\mathbb{C}^\times$-invariants

$t \longrightarrow t^2$

bosonic deg 2

Cat of BCs in the B-twist

$$\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma)) \otimes_{\mathrm{Vect}} \mathbb{C}((t))\text{-mod}$$

~~algebra of local ops~~ becomes $\mathcal{O}(\mathfrak{h}^*[2]/W)((t))$

In particular $\mathrm{Spec}\, H^0(\mathcal{O}(\mathfrak{h}^*[2]/W)((t)))$

$\simeq \mathfrak{h}^*/W$

Conjecture

Cat compatible with $v \in \mathfrak{h}^*/W$

$$\simeq \mathrm{IndCoh}_{\mathcal{N}_{L_v}}\left(\mathrm{Flat}_{L_v}(\Sigma)\right) \otimes_{\mathrm{Vect}} \mathbb{C}((\epsilon))\text{-mod}$$

where L_v is the stabilizer of v

We get sheaves on $\mathrm{Flat}_G(\Sigma)$

sing supported on $\mathrm{Arth}^v_G(\Sigma) = \mathrm{Arth}_G(\Sigma) \times_{\mathfrak{h}^*/W} \{v\}$

In fact

$$\mathrm{Arth}_{L_v}(\Sigma)^{\wedge}_{\mathcal{N}_{L_v}} \cong \mathrm{Arth}_G(\Sigma)^{\wedge}_{\mathrm{Arth}^{\vee}_G(\Sigma)}$$

Question: What about the A-side?

According to Arinkin–Gaitsgory

$$\mathrm{IndCoh}(\mathrm{Flat}_{\check{G}}(\Sigma)) \simeq \text{“}D\text{-mod}^{\text{ren}}(\mathrm{Bun}_G(\Sigma))\text{”}$$

renormalized D-modules