Recap

- Can model classical field theories by (-1)-symplectic stacks $\mathcal{M}$
- Procedure for building new theories from old "twisting", assuming $\mathcal{M}$ formal over a base $\mathcal{B}$, for us $\mathcal{B} = Bun_G(X)$
- Depends in particular on an odd symmetry Q, $Q^2 = 0$ generator of $\Pi\mathbb{C}$

Can find such a Q from an action of supersymmetry algebra

Exam...

N=...

Example

$N=4$ Super Yang Mills is a theory with an action of

$$(SO(4) \ltimes \mathbb{C}^4) \ltimes \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

choose Q_{hol} in $S_+ \otimes W$ $\quad Q_{hol}^2 = 0$

$S_+ = \langle a_1, a_2 \rangle$

$S_- = \langle b_1, b_2 \rangle$

then **Theorem** (E–Yoo)

the twist wrt Q_{hol} on a compact ... is equiv to

$Q_{hol} = a \otimes w$

$$T^*[-1]\, \mathrm{Higgs}_G(X)$$

holomorphically twisted $N=4$

$$\mathrm{Bun}_G(\mathbb{CP}^{3|4}) = \mathrm{Maps}(\mathbb{CP}^{3|4}, BG)$$

f $\Pi\mathbb{C}$

algebra

$$\mathcal{M}_{hol}(X) = T[1]\,\mathrm{Maps}(T[1]X, BG)$$

shifted tangent complex to $\mathcal{M}_{hol}(X)$ at a G-bundle P on X

$$= \left(\Omega^{*,*}(X, \mathfrak{g}_P)[1] \oplus \Omega^{*,*}(X; \mathfrak{g}_P)[2], (\bar{\partial}\right.$$

Two deformations we can consider

1) Can turn on ∂ on the two factors

2) Can turn on an iso between the two factors

For general X

deform $T[1]X$ to X_{dR}

family over $\mathbb{A}^1$

Fiber over 0 is $T[1]X$

Fiber over 1 is X_{dR}

(Fiber over λ is $X_{\lambda\text{-}dR}$)

ringed spaces $(X, (\Omega^{*,*}(X), \bar\partial))$

deform $\bar\partial$ to $\bar\partial + \lambda\partial$

Theorem (E-Yoo)

There's a family of classical field theories over $\mathbb{C}^2$ whose fiber over (λ, μ) is $\mathrm{Maps}(X_{\lambda\text{-}dR}, BG)_{\mu\text{-}dR}$. This coincides with the family of twists of ($N=4$ SYM / $\mathcal{M}_{\mathrm{Bun}}(X)$) defined by Kapustin-Witten.

at $(1,0)$ get $T^*[-1]\,\mathrm{Maps}(X_{dR}, BG) = T^*[-1]\,\mathrm{Flat}_G(X)$

at $(0,1)$ get $\mathrm{Higgs}_G(X)_{dR}$

Local Story

Claim B-twisted theory assigns to a compact submanifold Y of X the space

$$T^*[k-1]\mathrm{Flat}_G(Y) \qquad k = \mathrm{codim}\, Y$$

sections of tangent cx near Y

$$= \Omega^\cdot(Y, \mathfrak{g}_P)[1] \oplus \Omega^\cdot(Y, \mathfrak{g}_P)[2]$$

$$T[1]\mathrm{Flat}_G(Y)$$

$X) \rightsquigarrow$ B-twist

$\sim$ A-twist

In particular, if $Y = pt$

$$\mathcal{M}_B(pt) \cong T^*[3]BG$$

$$\simeq \mathfrak{g}^*[2]/G$$ adjoint quotient

Definition: Local classical observables are functions on local sol's to eq'ns of motion

Here $\mathrm{Obs}^{cl}_B \cong \mathcal{O}(\mathfrak{g}^*[2]/G) \cong \mathcal{O}(\mathfrak{h}^*[2]/W)$

affinization

What structure does this have?

In general, in topological field theories of dim n (eg twists wrt topological supercharges), local observables have the structure of a <u>$\mathbb{P}_n$-algebra</u> — cdga A with graded Lie bracket

$$A \otimes A \longrightarrow A[1-n]$$

which is a graded derivation for the product

<u>Eg</u> n-shifted derived stack X, $\mathcal{O}(X)$ is always a $\mathbb{P}_{n+1}$-algebra

<u>Here</u> $n=3$

$$\mathfrak{g}^*[2]/G = T^*[3]BG \quad 3\text{-symplectic}$$

So its functions form a $\mathbb{P}_4$-algebra

In fact this bracket is trivial for degree reasons

Quantization $n \geq 2$

In general, in this field theoretic top'l context quantizations of local observables are in particular

$\mathbb{E}_n$ deformations of the $\mathbb{P}_n$-algebra.

Are there non-trivial $\mathbb{E}_4$-deformations of $\mathcal{O}(\mathfrak{g}^*[2]/_{G})$?

No. $\mathbb{E}_4$-deformations are controlled by 3-shifted Polyvector fields (Toën)

$$\mathrm{Pol}(\mathfrak{g}^*[2]/G, 3)[4] \rightsquigarrow \text{dg Lie algebra}$$

Concentrated in even degree $\Rightarrow$ no deg 1 els, so soln to Maurer-Cartan

Intro to Geometric Langlands

<u>Idea</u> "Categorical non-abelian Fourier transform"

Roughly. $\mathrm{Bun}_G(\Sigma)$ $\quad \Sigma$ smooth closed curve

G reductive $\mathbb{C}$ group

Decompose D-modules on $\mathrm{Bun}_G(\Sigma)$ in terms of nice "eigensheaves for natural operators"

en)

o deg 1 els,

o solutions to

Maurer-Cartan

The dual space (indexing eigenvalues)

is $\mathrm{Flat}_{G^\vee}(\Sigma)$

$G^\vee$ Langlands dual group

<u>Conjecture</u> ("Best hope")

There is an equivalence

$$D\text{-mod}(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{QCoh}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

"eigensheaves" $\longmapsto$ skyscraper at that eigenvalue

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Mnemonic $G = GL_n = G^\vee$

"Category of flat bundles on space of vector bundles $\simeq$ Category of vector bundles on space of flat bundles"

e.g. · G abelian Conjecture is a theorem (Laumon, Rothstein)
it's literally a Fourier (-Mukai) transform

· G non-abelian, Conjecture is false as written
even if $\Sigma = \mathbb{P}^1$ (V Lafforgue)

Conjecture (Arinkin–Gaitsgory)

$$\text{D-mod}(\text{Bun}_G(\Sigma)) \simeq \text{IndCoh}_{\mathcal{N}}(\text{Flat}_{G^\vee}(\Sigma))$$

Ind-completion of cat of coherent sheaves w/ "nilpotent singular support" — Tomorrow

KW claimed category assigned to a curve Σ in

- B-twist group $G^\vee$: $\text{QCoh}(\text{Loc}_{G^\vee}(\Sigma))$ — moduli of $G^\vee$-local systems

IR S-duality

- A-twist group G: $\text{D-mod}(\text{Bun}_G(\Sigma))$

Story Today

Propose an ansatz

if moduli space assigned to a curve Σ is a shifted cotangent $T^*[1]W(\Sigma)$

$\mathcal{M}(\Sigma)$

say Category of boundary conditions on Σ

is $\mathrm{IndCoh}(W(\Sigma))$

"categorical geometric quantization"

This gives us

$$\mathcal{M}_B(\Sigma) \simeq T^*[1]\,\mathrm{Flat}_G(\Sigma)$$

get the right alg structure

$\Rightarrow$ Cat of BCs $=$ $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$

Can also check

$$\mathcal{M}_A(\Sigma) \simeq T^*[1]\left(\mathrm{Bun}_G(\Sigma)_{dR}\right)$$

$\Rightarrow$ Cat of BCs $=$ $\mathrm{IndCoh}\left(\mathrm{Bun}_G(\Sigma)_{dR}\right) \simeq D\text{-mod}\left(\mathrm{Bun}_G(\Sigma)\right)$