An Algebraic Introduction to Kapustin-Witten Theory

Kapustin-Witten

Gauge theory $\longleftrightarrow$ Geometric Langlands

Motivation

1) G

2) N

Motivation

1) Geometric Langlands

involves algebraic structures on moduli spaces whose origin is unclear from gauge theory

2) Nice example of a general formalism for classical/quantum field theory

Outline

Lecture 1) Wha

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gauge theory

al formalism

Field theory

Outline

Lecture 1) What is KW theory?

"Top^l Twists of Gauge Theory" - what does this mean?

First: Intro to derived symplectic geometry

based on joint work
with
Philsang Yoo

2) Local phenomena

algebra assigned to
the ball "local observables"

Category assigned to an
alg curve ~ connection to
geom Langlands

3) Origin of "Singular support conditions"
in GL from the action of local observables

Derived geometry & shifted symplectic geometry

A derived stack is a functor

$$X \; cdga^{\leq 0} \longrightarrow sSets$$

commutative dgas in degrees ≤ 0 — simplicial sets

satisfying a descent condition for étale topology

Examples

1) Usual schemes or stacks are examples of derived stacks.

2) simplicial sets are examples constant functor valued in Z (Sometimes called Z_{Betti})

3) S commutative dga in deg ≤ 0

$$\mathrm{Spec}(S)(R)_n = \mathrm{Hom}_{cdga}(S, R \otimes \Omega_{dg}(\Delta_n))$$

affine ~~derived~~ scheme

there is a ...

$\underline{\mathrm{Map}}(X$

5) For a d

4) If X & Y are derived stacks there is a mapping Stack

$$\underline{Map}(X,Y)(R) = Hom_{dStacks}(X \times Spec(R), Y)$$

5) For a derived stack X & $k \in \mathbb{Z}$ There's a shifted tangent space $T[k]X$

$$Map(Spec\, \mathbb{C}[\varepsilon], X)$$

where $|\varepsilon| = -k$

$$(S, R \otimes \Omega_{dg}(\Delta_n))$$

6) X derived the de R[…] of X is $X_{dR}(R$[…]

$\ldots$ Spec(R), Y)

$k \in \mathbb{Z}$

$T[k]X$

$=$

$Map(Spec\, \mathbb{C}[\varepsilon], X)$

where $|\varepsilon| = -k$

6) X derived stack, the <u>de Rham stack</u> of X is

$X_{dR}(R) = X(R^{red})$

Symplectic Structures

For our examples, any X will admit a cotangent complex $\mathbb{L}_X$ quasi-coherent sheaf on X

dual tangent complex $\mathbb{T}_X$

Definition

A p

Definition

A p-form of degree k on X is a section of degree k (map from $\mathcal{O}_X[-k]$) of

$$\Omega^p_X = \mathrm{Sym}^p(\mathbb{L}_X[1])[-p]$$

Have a map $\Omega^\bullet_X \xrightarrow{d_{dR}} \Omega^{\bullet+1}_X[1]$ extending derivative $\mathcal{O}_X \to \mathbb{L}_X$ as a derivation

d_{dR} degree -1 (internal degree)
weight 1 (p-form degree)

Definition

$\Omega^\bullet_{cl,X}$

$\mathbb{Q}_\ell[-n])$

Definition

$$\Omega^\bullet_{\mathbb{G}_a, X} = (\Omega^\bullet_X \otimes_{\mathbb{C}} \mathbb{C}[[u]],$$

$d_{dR} + d_{dR}^{u})$

u parameter of degree 2

s a derivation

Examples

1) $T^*[k]X$ always has a k-shifted symplectic structure

2) If G complex semisimple group, BG admits a 2-shifted symplectic structure

$\mathbb{T}_{BG} \simeq \mathfrak{g}[1]$ with adjoint G-action

2-shifted symplectic structures $\cong$ invariant non-degenerate pairings

$\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$

$\mathbb{L}_{BG} \simeq \mathfrak{g}^*[-1]$

4)

3) Theorem (PTVV)

There's a notion of a k-shifted Lagrangian structure on a map $L \xrightarrow{f} X$

If $L_1 \xrightarrow{f_1} X$ $L_2 \xrightarrow{f_2} X$ are k-shifted Lag^n then $L_1 \times_X L_2$ is $(k-1)$-shifted symplectic

4) <u>Theorem</u> (PTVV, AKSZ construction)

IF X is k-shifted symplectic

& M is "Compact n-oriented"

roughly, there's a degree n perfect pairing on its cohomology

Then

$\mathrm{Maps}(M, X)$

is $(k-n)$-shifted symplectic

e.g. $M = M_{Betti}$ for a smooth oriented n-manifold

or $M = Y_{dR}$ Y a proper smooth variety $\dim_{\mathbb{C}} = n$

Or M smooth Calabi-Yau variety $\dim_{\mathbb{C}} = n$

...d symplectic

Classical Field Theory

A classical field theory is modelled by a space Φ of fields & an action functional $\Phi \xrightarrow{S} \mathbb{C}$

classical phase space = critical points of S.

Definition

The de

Definition

The derived critical locus of S is the derived intersection

$$dCrit(S) = \underset{\text{zero section}}{\Phi} \cap \underbrace{\Gamma_{dS}}_{\text{graph of } dS} \subset T^*\Phi$$

S. $\Phi \to T^*\Phi$

& $\Gamma_{dS} \to T^*\Phi$ are 0-shifted Lag^n

so $dCrit(S)$ is (-1)-shifted symplectic

The "twist" ...
is the total ...

Definition

A classical field theory on M is
a (-1)-shifted symplectic stack

Twisting

locally if a $\mathbb{Z}/2$-graded cochain complex V
has an action of $\mathbb{C}^\times \ltimes \Pi\mathbb{C} = \mathrm{Aut}(\Pi\mathbb{G}_a)$

weight 1 / fermionic/odd degree / $\mathbb{C}^\times$-weight / action of $\Pi\mathbb{C}$

equivalently V can be given an additional $\mathbb{Z}$-grading & diff^l

The "twist" of V wrt this action
is the total complex wrt two gradings on structures

This construction appears in physics $V = \Pi_{\alpha c k(s)}[-1]$

$\mathbb{C}^\times \ltimes \Pi\mathbb{C}$ - action comes
from a supersymmetry action

<u>Claim</u> Can build examples of classical field
theories by this construction

$(\mathbb{G}_a)$

$\mathbb{C}^\times$-weight

odd in of $\Pi\mathbb{C}$

ds diff^l

Kapustin-Witten Theory

Example of this construction starting from "N=4 Super Yang-Mills theory"

admits an algebraic construction

i.e. it can be modelled by a (-1)-symplectic derived stack

It has

Defin

It has an action of "$N=4$ supersymmetry algebra"

Definition (complex $N=4$ 4d SUSY algebra)

$$\Big(\underbrace{so(4,\mathbb{C}) \ltimes \mathbb{C}^4}_{\text{cx Poincaré}} \times \underbrace{sl(4,\mathbb{C})}_{\text{"R-symmetries"}}\Big) \ltimes \overbrace{\Pi}^{\text{odd degree}}\overbrace{(S_+ \otimes W \oplus S_- \otimes W^*)}^{\text{supersymmetries}}$$

$S_\pm$ semi spin reps of $so(4,\mathbb{C}) \cong sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$

W fundamental rep of $sl(4,\mathbb{C})$

$$\Gamma\ (S_+ \otimes W) \otimes (S_- \otimes W^*) \longrightarrow \mathbb{C}^4$$

given by $S_+ \otimes S_- \xrightarrow{\sim} \mathbb{C}^4$

plus $W \otimes W^* \to \mathbb{C}$

KW

Use SUSY action to construct $\mathbb{C}^\times \ltimes \Pi\mathbb{C}$-actions

$\Pi\mathbb{C}$ actions come from square 0 odd elts of SUSY algebra

i.e. Q with $[Q,Q]=0$

fermionic part $S_+ \otimes W \oplus S_- \otimes W^*$

<u>Definition</u>

Such a Q is called <u>topological</u> if $[Q,-]\ S \longrightarrow \mathbb{C}^4$ is surjective

<u>holomorphic</u> if image of $[Q,-]$ is $\frac{1}{2}$ dimensional

$s \otimes w^*$

is surjective

dimensional

KW theories are topological twists of $N=4$ super Yang-Mills theory

One can do this twisting construction globally on derived stacks in a nice situation. Say $d\mathrm{Crit}(S) = \mathcal{M}$

Suppose we have maps

π, σ are quasi-isomorphisms

isos on H^0 of the derived stacks

$$\begin{array}{c} \mathcal{M} \\ {\scriptstyle\pi}\downarrow \uparrow{\scriptstyle\sigma} \\ \mathcal{B} \end{array}$$

σ section of π

Suppose $\mathbb{C}^\times \ltimes \Pi\mathbb{C}$ acts on $\mathcal{M}$

equivariantly for trivial action on $\mathcal{B}$

eg $\mathcal{B} = \mathrm{Bun}_G(Y)$ ~ $\mathcal{M}$ is an "algebraic gauge theory"

Example (holo twisted $N=4$ theory)

X compact complex surface

$\mathcal{M} = T^*[-1]\,\mathrm{Higgs}_G(X)$

classically G-bundle P plus a section of $\mathfrak{g}_P \otimes K_X$ satisfying a condition

$\mathrm{Map}(T[1]X, BG)$

"Dolbeault Stack of

$\mathrm{Higgs}_G(X)$ is (-2)-shifted symplectic

Can rewrite this as

$$T[1]\,\mathrm{Higgs}_G(T[1]X, BG)$$

Fact there are two twists deforming two copies of $T[1]$ – to $(-)_{dR}$

$\otimes K_X$