

# An Algebraic Introduction to Kapustin-Witten Theory

Kapustin-Witten

Gauge theory  $\longleftrightarrow$  Geometric  
Langlands

Motivation

1)  $G$

2)  $M$

## Motivation

### 1) Geometric Langlands

involves algebraic structures  
on moduli spaces whose  
origin is unclear from gauge theory

2) Nice example of a general formalism  
for classical/quantum field theory

## Outline

Lecture 1)

What  
To  
T

Outline

Lecture 1)

What is Kuranishi?

"Top<sup>l</sup> Twists of Gauge Theory" - what does this mean?

First. Intro to derived symplectic geometry

gauge theory

real formal field theory

## 2) Local phenomena

Algebra assigned to  
the ball "local observables"

Category assigned to an  
alg curve ~ connection to  
geom Langlands

## 3) Origin of "Singular support conditions" in GL from the action of local observables

based on joint work  
with  
Philsang Yoo

# Derived geometry & shifted symplectic geometry

A derived stack is a functor

$$X \text{ cdga}^{\leq 0} \rightarrow \text{sSets}$$

Commutative dgas  
in degrees  
 $\leq 0$

Simplicial  
Sets

Satisfying a descent condition for étale topology

## Examples

1) Usual schemes or stacks are examples of derived stacks.

2) Simplicial sets are examples of constant functors valued in  $\mathcal{Z}$  (Simplicies called  $\mathcal{Z}^{\text{Betti}}$ )

3)  $S$  commutative dga in  $\text{deg} \leq 0$

$$\text{Spec}(S)_{(R)_k} = \text{Hom}_{\text{dga}}(S, R \otimes \Omega_{\text{dg}}(\Delta_k))$$

diff. derived scheme

there is a

Map (X

5) For a d

4) If  $X$  &  $Y$  are derived stacks  
there is a mapping stack

$$\underline{\text{Map}}(X, Y)(R) = \text{Hom}_{\text{stacks}}(X \times \text{Spec}(R), Y)$$

5) For a derived stack  $X$  &  $k \subset \mathbb{Z}$

There's a shifted tangent space  $T[k]X$

$$(S, R \otimes \Omega_{d_g}(\Delta_k))$$

$$\text{Map}(\text{Spec}(\mathbb{C}[\epsilon]), X)$$

where  $|\epsilon| = -k$

6)  $X$  derived  
the de Rham  
of  $X$   
is  $X_{dR}(R)$

$\times \text{Spec}(R), Y)$

6)  $X$  derived stack,  
the de Rham stack  
of  $X$

$$\Rightarrow X_{\text{dR}}(R) = X(R^{\text{red}})$$

$k \subset \mathbb{Z}$   
 $T[k] X$

$\text{Map}(\text{Spec}(\mathbb{Z}[t]), X)$

where  $|E| = -k$



## Symplectic Structures

For our examples, any  $X$   
will admit a cotangent complex

$\mathbb{L}_X$  quasicoherent sheaf on  $X$

dual tangent complex  $\mathbb{T}_X$

Definition

A

## Definition

A  $p$ -form of degree  $k$  on  $X$   
is a section of degree  $k$  (map from  $\mathcal{O}_X[-k]$ )  
of

$$\Omega_X^p = \text{Sym}^p(\mathcal{L}_X[1])[-p]$$

Have a map  $\Omega_X^\bullet \xrightarrow{d_{\text{DR}}} \Omega_X^{\bullet+1}[1]$   
extending derivative  $\mathcal{O}_X \rightarrow \mathcal{L}_X$  as a derivation

dim degree  $-1$  (internal degree)  
weight  $1$  (p-form degree)

## Definition

$$\Omega_{\text{cl}, X}^\bullet$$

Definition

$\mathbb{Q}_x[-k]$

$$\Omega_{a,x}^{\bullet} = (\Omega_x^{\bullet} \otimes_{\mathbb{C}} \mathbb{C}[u])$$

$$d_{a,x} = d_{a,x}^{\bullet}$$

$u$  parameter of degree 2

$\rightarrow$  a derivation

## Examples

1)  $T^*[k]X$  always has a  $k$ -shifted symplectic structure

2) If  $G$  complex reductive group,  $BG$  admits a  $\mathbb{Z}$ -shifted symplectic structure  $\mathbb{T}BG \cong \mathfrak{g}[1]$  with adic  $G$ -action

$\mathbb{Z}$ -shifted symplectic structures  $\cong$  invariant non-degenerate pairings

$$\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$$

$$\mathbb{T}BG \cong \mathfrak{g}[1]$$

3) Theorem (PTU)

There's a notion of a  $k$ -shifted  
Lagrangian structure on

a map  $L \rightarrow X$

If  $L_1 \rightarrow X$   $L_2 \rightarrow X$  are  $k$ -shifted

then  $L_1 \times_X L_2$  is  $(k-1)$ -shifted symplectic

4) Theorem (PTW, AKSZ ansatz)

IF  $X$  is  $k$ -shifted symplectic

&  $M$  is "Compact  $n$ -oriented"

roughly, there's a degree  $n$  perfect pairing on  
its cohomology

Then  
 $\text{Maps}(M, X)$   
is  $(k-n)$ -shifted  
symplectic

e.g.  $M = M_{\text{Bott}}$  for a smooth oriented  
 $n$ -manifold

or  $M = Y_{\text{dR}}$  for a proper smooth variety  $\dim_{\mathbb{C}} = n$

or  $M$  smooth Calabi-Yau variety  $\dim_{\mathbb{C}} = n$

# Classical Field Theory

Definition

The de

A classical field theory is  
modelled by a space  $\Phi$  of fields  
& an action functional  $\Phi \xrightarrow{S} \mathbb{R}$   
classical phase space = critical points of  $S$ .

## Definition

The derived critical locus of  $S$   
is the derived intersection

$$d(\text{Crit}(S)) = \Phi \cap T_{dS}$$

$$\begin{array}{c} \uparrow T^*\Phi \\ \Phi \end{array} \left. \vphantom{\begin{array}{c} \uparrow T^*\Phi \\ \Phi \end{array}} \right\} \text{graph of } dS$$

$$\Phi \rightarrow T^*\Phi \quad \text{zero section}$$

$$\& T_{dS} \rightarrow T^*\Phi \text{ are } 0\text{-shifted Lag}^{\vee}$$

so  $d(\text{Crit}(S))$  is  $(-1)$ -shifted symplectic



The lowest is the total

Definition

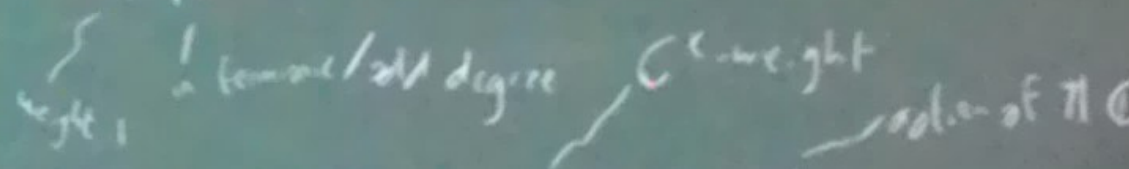
A classical field theory  $\mathcal{L}$  on  $M$  is a  $(-1)$ -shifted symplectic stack

Twisting

$\mathbb{Z}/2$ -graded

locally if a  $\wedge$ -cochain complex  $V$

has an action of  $\mathbb{C}^\times \times \mathbb{Z}/2 \times \mathbb{C} = \text{Aut}(\mathbb{Z}/2)$



equivalently  $V$  can be given an additional  $\mathbb{Z}$ -grading by diff<sup>k</sup>

The "twist" of  $V$  with this action  
is the total complex w/ two exact on structure.

This construction appears in physics  $V = \Pi_{\text{diffeo}}[-1]$   
 $\mathbb{C} \propto \pi \mathbb{C}$  - action comes

Clam Can build examples of classical field  
theories by this construction

(ex)

$\mathbb{C}$ -weight  
radius of  $\pi \mathbb{C}$   
& diff<sup>1</sup>

## Kapustin-Willen Theory

Example of this construction starting  
from "N=4 Super Yang-Mills theory"

{ admits an algebraic construction

ie it can be modelled by a (+)-Symplectic  
derived stack

Defin

It has an action of  $N=4$  supersymmetry algebra

Definition (complex  $N=4$   $\mathcal{N}=4$  susv algebra)

$$(so(4, \mathbb{C}) \ltimes \mathbb{C}^4 \times sl(4, \mathbb{C})) \ltimes \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

$\ltimes$  Parabolic       $\ltimes$  R-symmetry       $\Pi$  odd degree supersymmetric

$S_{\pm}$  Sem spin reps of  $so(4, \mathbb{C}) \cong so(2, 1) \oplus so(2, 1)$   
 $W$  Fundamental rep of  $sl(4, \mathbb{C})$

$$\Gamma(S_+ \otimes W) \otimes (S_- \otimes W^*) \rightarrow \mathbb{C}^4$$

given by  $S_+ \otimes S_- \cong \mathbb{C}^4$   
 plus  $W \otimes W^* \rightarrow \mathbb{C}$

Use SUSY action to construct  $\mathbb{C}^x \times \Pi\mathbb{C}$  -reduces

$\Pi\mathbb{C}$  reduces come from square 0 odd elts of SUSY algebra  
 $\dots Q$  with  $[Q, Q] = 0$

Definition

$S_{\text{red}} = Q$  is called topological if  $[Q, -] \hookrightarrow \mathbb{C}^4$  is surjective

holomorphic if range of  $[Q, -]$  is  $\frac{1}{2}$  dimensional

fermionic part  $S_{\text{ferm}} \otimes S_{\text{bos}}$

KW

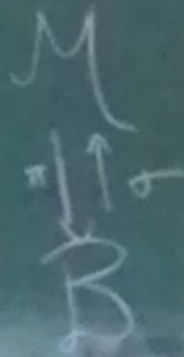
KW theories are topological twists of  
 $N=4$  super Yang-Mills theory

One can do this twisting construction  
globally on doped stacks in a nice  
situation. Say  $d(C, (CS)) = M$

Suppose we have maps

$\pi, \sigma$  are  
isomorphisms

isom on  $H^0$  of  
for doped stacks



$\sigma$  section of  $\pi$

Suppose  $\mathbb{C}^* \times \pi \mathbb{C}$  acts on  $M$   
equivalently for trivial action on  $B$

e.g.  $B = \text{Bun}_G(Y)$  -  $M$  is an "algebraic gauge theory"

Example (holomorphic twisted  $N=4$  theory)

•  $X$  compact complex surface

$$M = T^*[X] \text{ Higgs}_G(X)$$

classically  
is a section of  $\mathcal{G}_P \otimes K_X$   
satisfying a condition

$$\text{Map}(T^*[X], BG)$$

"Deligne Stack of

$\text{Higgs}_G(X)$  is  $(-2)$ -shifted symplectic

Can rewrite this as

$$T[1] \text{Higgs}_G(T[1]X, BG)$$

Fact there are two twists  
deforming two copies  
of  $T[1]-$

to  $(-)_DR$

$\otimes K_X$