An Algebraic Introduction to Kapustin-Witten Theory

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Abstract

Kapustin and Witten constructed a family of "twisted $N = 4$ gauge theories" in four dimensions in order to build a bridge between gauge theory and the geometric Langlands correspondence. In these lectures I'll introduce $N = 4$ theories and explain how to derive their twists in a purely algebraic way. I'll discuss some aspects of the quantization of these theories, and explain an application to the theory of singular supports in geometric Langlands. This is joint work with Philsang Yoo.

1 Lecture 1 – Kapustin-Witten Twists: a Global, Classical Description

1.1 Introduction

My goal with this series of lectures is to explain how the family of topological field theories constructed by Kapustin and Witten [KW07] can be mathematically understood in the setting of derived algebraic geometry. This is a worthwhile exercise for at least two reasons. On the one hand, Kapustin and Witten's work was motivated by the goal of drawing a bridge between supersymmetric gauge theory and the geometric Langlands conjecture – by carefully mathematically modelling these theories we can make this connection more precise and fill in gaps in the physical story, for instance by explaining the appearance of algebraic structures on moduli spaces, and singular support conditions in the modern formalulation of geometric Langlands. Similarly the physical approach suggests new structures in the geometric Langlands story itself which we'll discuss in the third lecture. On the other hand, Kapustin-Witten theory provides an interesting but computationally tractable example of the general formalism of classical and quantum field theory – for instance it's close to (but not fully) topological – which we can use as a test example for the more general problem of puzzling out the non-perturbative structure of similar twisted gauge theories in other dimensions. The work I'll talk about today is joint with Philsang Yoo [EY15, EY17]. Some parts of these notes are excerpted from those two papers.

I'll start with a brief summary of the structure of the three talks.

1. In the first lecture I'll talk about what it means to have a *topologically twisted field theory* from the point of view of derived algebraic geometry. This is, I claim, a natural setting in which to talk about classical field theory "non-perturbatively", that is, including the full data of the fields and action functional of a classical field theory. In order to get there, I'll start with an introduction to the facts about derived algebraic and shifted symplectic geometry that we'll refer to in the three talks, and then sketch a definition of what I mean by a classical field theory.

   Now, from this point of view to *twist* a classical field theory mean essentially to take the derived invariants with respect to an odd symmetry. I'll talk about a nice family of field theories – a certain kind of gauge theory which makes sense algebraically – in which we can perform this procedure, and tell you how to fit the famous physical $N = 4$ 4d gauge theory into this family. We'll then describe the Kapustin-Witten twists of this theory, and see some interesting representation theoretic moduli spaces appearing.

2. In the second lecture I'll talk about the local structure of these twisted field theories. I'll compute the local observables in these theories and see what happens when we try to quantize them – in fact these local observables do not admit any quantum corrections. We get something more interesting when we look at what these field theories assign to an algebraic curve. I'll explain an ansatz out of which we obtain the geometric Langlands categories by geometric quantization. I'll also give a short introduction to what exactly geometric Langlands theory is.

3. In the third and final lecture I'll explain the results of my paper with Philsang Yoo [EY17]. By investigating the action of local observables on the categories of boundary conditions on a curve, and imposing a condition on the support with respect to this action, we'll see that the singular support conditions of Arinkin and Gaitsgory [AG12] naturally arise. We can motivate these conditions physically, and generalize them to see some potentially interesting new structures on the geometric Langlands correspondence.

1.2 Some Facts About Derived Symplectic Geometry

In these talks I'm going to be talking about some examples of derived stacks with shifted symplectic structures – these things naturally appear in classical field theory. As such I'll start with a quick introduction to what these words mean. The field of derived algebraic geometry includes a lot of sophisticated theory (most of which I know very little about), but today I'm only going to state a few definitions along with one key theorem about shifted symplectic structures on mapping spaces. The theory of derived symplectic geometry was developed by Pantev, Toën, Vaqiué and Vezzosi [PTVV13] and we refer to their work for details. Whenever I say "category" I'll really mean "∞-category", but luckily for what I have to say today we won't need to engage with any of those issues.

So what is a derived stack? I'll give a sketch definition so that you have something to hold on to, but much more importantly I'll give a list of examples.

Definition 1.1. A *prestack* is a functor $X\colon \mathrm{cdga}^{\leq 0} \to \mathrm{sSet}$ from commutative dgas over $\mathbb{C}$ concentrated in degrees ≤ 0 to simplicial sets. A *derived stack* is a prestack that satisfies a descent condition for the étale topology.

Remark 1.2. For us all derived stacks will be Artin stacks locally of finite presentation. This is a technical condition that, for instance, ensures the tangent complex is perfect, hence dualizable, which will be important when we talk about symplectic structures.

1. Every classical scheme or stack is an example of a derived stack.

2. Every simplicial set M is an example of a derived stack by taking the *constant* functor $M(R) = M$.

3. If S is a commutative dga concentrated in degrees ≤ 0 then there is a derived stack called the *spectrum* of S whose R points are given by
$$\mathrm{Spec}(S)(R)_k = \mathrm{Hom}_{\mathrm{cdga}}(S, R \otimes \Omega^\bullet_{\mathrm{alg}}(\Delta_k)).$$
There's an alternative model for these derived schemes as ringed spaces $(X, \mathcal{O}_X)$, where $(X, H^0(\mathcal{O}_X))$ is a classical scheme and $H^i(\mathcal{O}_X)$ is a quasi-coherent sheaf for each $i > 0$.

4. If X and Y are derived stacks then there is a *mapping stack* $\underline{\mathrm{Map}}(X,Y)$ whose R points are given by
$$\underline{\mathrm{Map}}(X,Y)(R)_k = \mathrm{Hom}_{\mathrm{dStacks}}(X \times \mathrm{Spec}(R \otimes \Omega^\bullet_{\mathrm{alg}}(\Delta_k)), Y).$$

5. If X is a derived stack and k is an integer then there's a derived stack called $T[k]X$ – this k-shifted tangent space of X. We can model it as the mapping stack $T[k]X = \underline{\mathrm{Map}}(\mathrm{Spec}\,\mathbb{C}[\varepsilon], X)$ where ε is a degree $-k$ parameter satisfying $\varepsilon^2 = 0$.

6. If X is a derived stack then one can define the *de Rham stack* of X by
$$X_{\mathrm{dR}}(R) = X(R_{\mathrm{red}}),$$
where R_{red} is the quotient of R by its nilradical. Roughly speaking one should think of X_{dR} as what you get by identifying "infinitesimally close points" of X.

All of the derived stacks I'll talk about this week will be built using these examples and constructions.

Definition 1.3. We can talk about *quasi-coherent sheaves* on a derived stack X by taking the limit over all affine derived schemes mapping in, where we set $\mathrm{QC}(\mathrm{Spec}\, R) = R\text{-mod}$. An important examples is gives by the *cotangent complex* $\mathbb{L}_X$ of a derived stack X. For the examples we'll be working with, the cotangent complex is always dualizable; we call its dual the *tangent complex* $\mathbb{T}_X$.

Once we have the notion of the cotangent complex, it makes sense to talk about p-forms on a derived stack. However, since the tangent complex is a sheaf of cochain complexes, likewise there will be a *complex* of p-forms.

Definition 1.4. A *p-form of degree k* on X is a section of degree k (meaning a map of sheaves from $\underline{C}[-k]$) of the sheaf of *p-forms* on X:

$$\Omega^p_X = \mathrm{Sym}^p(\mathbb{L}_X[1])[-p].$$

The *de Rham differential* is a sheaf map $\mathrm{d}_{\mathrm{dR}} \colon \Omega^\bullet_X \to \Omega^{\bullet+1}_X[1]$ which extends the derivative $\mathcal{O}_X \to \mathbb{L}_X$ as a derivation. The complex $\Omega^\bullet_X$ is therefore *bigraded*: there's an internal differential d of degree 1 coming from the differential on $\mathbb{L}_X$, and there's also the de Rham differential which has degree -1 (even though it increases the *weight* – the p above – by 1). A *closed p-form of degree k* is a section of degree k of the complex

$$\Omega^\bullet_{\mathrm{cl},X} = (\Omega^\bullet_X \otimes_{\mathbb{C}} \mathbb{C}[[u]], \mathrm{d} + u\mathrm{d}_{\mathrm{dR}})$$

where u is a parameter of degree 2. Note that in contrast to the classical story, being closed is not a condition. Indeed if you compute the k^{th} cohomology of $\Omega^\bullet_{\mathrm{cl},X}$ you find not a de Rham-closed element, but an element which is de Rham closed up to the addition of a d-exact form which in turn satisfies a tower of higher coherences. In particular there's a map $\Omega^\bullet_{\mathrm{cl},X} \to \Omega^\bullet_X$ but it's not injective in any sense.

With this in mind, the definition of a shifted symplectic structure is roughly what you might guess.

Definition 1.5. A *k-shifted symplectic structure* on a derived stack X is a closed 2-form of degree k which is *nondegenerate* in the following sense. Any 2-form of degree k defines a morphism $\mathbb{T}_X \to \mathbb{L}_X[k]$ and any closed 2-form defines an associated 2-form: the closed 2-form is non-degenerate if this induced morphism is a quasi-isomorphism.

There are a few ways of building shifted symplectic structures (see [PTVV13] for details and proofs).

1. The k-shifted cotangent space $T^*[k]X$ of a derived stack always carries a canonical k-shifted symplectic structure.
2. If G is a complex reductive group, the classifying stack BG carries a canonical 2-shifted symplectic structure. The tangent complex of BG is equivalent to $\mathfrak{g}$ placed in degree -1 and equipped with the adjoint action of G. One can verify that the closed 2-forms of degree 2 are exactly the G-invariant pairings $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, and non-degeneracy is equivalent to non-degeneracy of the pairing. In other words there's a canonical 2-shifted symplectic structure given by the Killing form.
3. This won't be too important for this week's lectures, but it's worth mentioning that there's a notion of a k-shifted *Lagrangian* structure on a morphism of derived stacks $f \colon L \to X$ when X is k-shifted symplectic (there's no restriction, by the way, that f is an embedding).

   Theorem 1.6. If $L_1 \to X$ and $L_2 \to X$ are k-shifted Lagrangian then the derived fiber product $L_1 \times_X L_2$ carries a canonical $(k-1)$-shifted symplectic structure.

4. We'll use the following fact a lot more.

   Theorem 1.7 (AKSZ-PTVV)**.** Let X be a k-shifted symplectic derived stack, and let M be a derived stack which is *compactly oriented* of degree n. Roughly speaking this is a perfect pairing on its cohomology, so for instance closed oriented n-manifolds are examples (by Poincaré duality), or n-dimensional smooth Calabi-Yau varieties (by Serre duality). Then there is a canonical $(k-n)$-shifted symplectic structure on the mapping space $\underline{\mathrm{Map}}(M, X)$.

I can give you an idea of roughly what this symplectic structure is: it's defined via *transgression*. There's a natural *evaluation* map $\mathrm{ev}\colon \underline{\mathrm{Map}}(M, X) \times M \to X$, so by pulling back the symplectic form on X we obtain a closed 2-form of degree k $\mathrm{ev}^*(\omega)$ on $\underline{\mathrm{Map}}(M, X) \times M$. The degree n compact orientation defines a pushforward map from degree k closed 2-forms on the product to degree $k - n$ closed 2-forms on $\underline{\mathrm{Map}}(M, X)$. One can then demonstrate that this closed 2-form is non-degenerate.

1.3 $N = 4$ Gauge Theory

1.3.1 Classical Field Theories

So, what is a classical field theory? If you attended my talk yesterday you heard one definition in terms of a sheaf of dg Lie algebras with an invariant pairing. The definition I'll give today will be a *global* version of that local definition. In physics, a Lagrangian field theory on a space M is modelled by a sheaf of spaces Φ, called the *fields*, along with a map S to the constant sheaf called the *action functional*. The classical states of this system are modelled by the critical locus of S. Passing to the critical locus loses some information about the pair (Φ, S), so let's instead use a derived version of this construction.

Definition 1.8. Given a derived stack Φ and a function $S\colon \Phi \to \mathbb{C}$, the *derived critical locus* of S is the derived intersection

$$\mathrm{dCrit}(S) = \Phi \cap_{T^*\Phi} \Gamma_{\mathrm{d}S},$$

where $\Phi \to T^*\Phi$ is the zero section, and $\Gamma_{\mathrm{d}S}$ is the graph of the 1-form $\mathrm{d}S$. Note that $T^*\Phi$ is 0-shifted symplectic and both the 0-section and the graph of a 1-form are Lagrangian, so by Theorem 1.6 the derived critical locus is always (-1)-shifted symplectic.

Remark 1.9. You might object that if we really want to consider the space of fields on a non-compact spacetime then the action functional is not typically well-defined (there's a well-defined Lagrangian density but it doesn't have a finite integral). This is not a problem for this formalism: even though S isn't necessarily well-defined its derivative $\mathrm{d}S$ still will be.

We can model a classical field theory by its full *derived* critical locus with its (-1)-shifted symplectic structure.

Definition 1.10. A *classical field theory* on M is a sheaf EOM (for the solutions to the *equations of motion*) of derived stacks on M with a (-1)-shifted symplectic structure on the global sections $\mathrm{EOM}(M)$.

Remarks 1.11.
1. A more sensitive definition might include a (-1)-shifted Poisson structure on the local sections $\mathrm{EOM}(U)$ compatible with the sheaf structure.
2. In general the full sheaf is often difficult to construct even without tracking the Poisson bracket. In what follows we'll jut study a single (-1)-shifted symplectic stack at a time, modelling the global sections.
3. To recover the local definition of a classical field theory that I discussed yesterday, one chooses a point in $\mathrm{EOM}(M)$ and takes the (-1)-shifted tangent complex. This is automatically a dg Lie algebra, and the shifted symplectic structure induces a pairing onthe shifted tangent complex of degree -3. This is what we call the BV-BRST complex of the classical field theory equipped with its antibracket.

1.3.2 $N = 4$ Super Yang-Mills

The examples we'll discuss today will be classical field theories defined as twists of a theory called $N = 4$ *super Yang-Mills theory* in 4-dimensions. Let me try to explain what this theory is and why it might have anything to do with algebraic geometry. There are two different ways of building this 4d gauge theory. The original construction uses $N = 1$ super Yang-Mills theory on $\mathbb{R}^{10}$. I won't talk about this approach today, although it's a fun calculation. Instead I'll describe another construction due to Penrose and Ward via *super twistor space* (for details we refer to the book [WW91] of Ward and Wells).

Definition 1.12. $N = 4$ super twistor space is a super complex algebraic variety whose bosonic part has dimension 3. It's actually quite simple, it's nothing but

$$\mathbb{PT}^{N=4} = \mathbb{P}(\mathbb{C}^{4|4}),$$

the space of (ordinary, bosonic) lines in a super vector space. Even more concretely it's the total space of the vector bundle $\Pi(\mathcal{O}(1)^{\oplus 4})$ on $\mathbb{CP}^3$. Super twistor space admits a map (sometimes called the *Penrose map*) down to the 4-sphere coming from the map

$$\mathbb{CP}^3 \to \mathbb{HP}^1 \cong S^4$$

whose fibers are the total space of an odd vector bundle over $\mathbb{CP}^1$ – the so-called *twistor lines* (exercise: check this). In particular there's a map

$$p\colon \mathbb{PT}^{N=4} \setminus \mathbb{CP}^1 \to \mathbb{R}^4$$

obtained by throwing out the fiber over ∞. Note: even though the left-hand side is a complex manifold, this map is not holomorphic for any complex structure on $\mathbb{R}^4$.

The *Berezinian* (the super version of the canonical bundle) of $\mathbb{PT}^{N=4}$ is trivializable. Indeed, one can compute the Berezinian of any superprojective space $\mathbb{CP}^{m|n}$ to be

$$\begin{aligned}\mathrm{Ber}(\mathbb{CP}^{m|n}) &\cong K_{\mathbb{CP}^n} \otimes \wedge^n(\mathcal{O}(1)^{\oplus n}) \\ &\cong \mathcal{O}(-m-1) \otimes \mathcal{O}(n) \\ &\cong \mathcal{O}(n-m-1)\end{aligned}$$

so in this case $m = 3, n = 4$ so the Berezinian is trivial.

$N = 4$ super Yang-Mills will arise as the pushforward along p of a classical field theory on super twistor space. Let's explain what that theory is.

Definition 1.13. One can define *algebraic Chern-Simons theory* on any super Calabi-Yau smooth super 3-fold, i.e. a smooth super variety Y whose bosonic part has dimension 3 with trivialized Berezinian. As a (-1)-shifted symplectic derived stack it's defined as

$$\mathrm{Bun}_G(Y) = \underline{\mathrm{Map}}(Y, BG).$$

This is (-1)-shifted symplectic whenever Y is compact by Theorem 1.7 since BG is 2-shifted symplectic and Y is 3-oriented.

Remark 1.14. This is the origin of the algebraic structures that we'll use in Kapustin-Witten twisted $N = 4$ theory: one usually says that one obtains 4d $N = 4$ theory as the pushforward of *holomorphic* Chern-Simons theory on super twistor space, i.e. considering the complex analytic stack of holomorphic G-bundles. We observe that this stack admits a natural algebraic structure which carries over to provide an algebraic structure to the Kapustin-Witten topological twists.

Remark 1.15. I'm lying a little bit: the theory obtains by pushing forward holomorphic Chern-Simons theory to $\mathbb{R}^4$ isn't exactly the same as $N = 4$ super Yang-Mills, it's only the *anti-self-dual sector* of that theory. Boels, Mason and Skinner [BMS07] demonstrated that the holomorphic Chern-Simons theory can be modified in order to produce the full $N = 4$ theory under compactification, but we won't need to worry about this distinction – the difference between the full $N = 4$ theory and the anti-self-dual $N = 4$ theory vanishes after performing the twist.

Remark 1.16. We run into trouble when we try to define untwisted $N = 4$ super Yang–Mills theory non-perturbatively via compactification along the twistor fibers, because the Penrose map p is *not* holomorphic for any complex structure on $\mathbb{R}^4$. As such, a Zariski open set $U \subseteq \mathbb{C}^2$ does not lift to a Zariski set $p^{-1}(U) \subseteq \mathbb{PT} \setminus \mathbb{CP}^1$. This is not a problem in the analytic setting; any open set in a complex manifold admits a canonical complex structure, but generally not an *algebraic* structure. It is not particularly surprising that we encounter such problems: there's no reason that a metric-dependent theory like untwisted $N = 4$ gauge theory should admit a description purely in terms of algebraic geometry. What we'll actually do is use this algebraic theory to define twisted versions of $N = 4$ which *will* make sense as theories on $\mathbb{R}^4$.

Remark 1.17. Another related issue is the fact that when we remove the twistor line over the point at ∞ the super twistor space is no longer compact, so we no longer have an AKSZ shifted symplectic structure (at most we'll have a shifted Poisson structure). Similarly, we won't really use the shifted symplectic structure for the untwisted $N = 4$ theory, only the structure we obtain after turning on a twist, at which point we can define the theory on more general compact complex surfaces and avoid this issue of non-degeneracy.

1.3.3 The $N = 4$ Supersymmetry Algebra

The reason this theory is special is that it carried an action of the $N = 4$ *supersymmetry algebra*. I talked about supersymmetry algebras in general yesterday, but I'll say now what this specific example looks like. For simplicity I'll just describe the complexified supersymmetry algebra – the data of its real form won't be so important.

Definition 1.18. There is an exceptional isomorphism $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$. The 4-dimensional complex spin representation of $\mathfrak{so}(4)$ is the sum $S_+ \oplus S_-$ of the fundamental representations of the two copies of $\mathfrak{sl}(2;\mathbb{C})$. The 4-dimensional fundamental representation of $\mathfrak{so}(4;\mathbb{C})$ is identified with the tensor product $S_+ \otimes S_-$.

The complexified $N = 4$ *supersymmetry algebra* is the complex super Lie algebra

$$\mathcal{A}^{N=4} = ((\mathfrak{so}(4;\mathbb{C}) \ltimes \mathbb{C}^4)) \times \mathfrak{sl}(4;\mathbb{C})) \ltimes \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

where W is a complex vector space of dimension 4. The algebra $\mathfrak{sl}(4;\mathbb{C})$ is the algebra of *R-symmetries*: it acts on W as the fundamental representation. There is a bracket between the two summands of the fermionic part valued in $\mathbb{C}^4$ by the evaluation pairing $W \otimes W^* \to \mathbb{C}$ and the canonical identification $S_+ \otimes S_- \cong \mathbb{C}^4$.

Remark 1.19. On the level of the supersymmetry algebra there is a full $\mathrm{GL}(4;\mathbb{C})$ of R-symmetries acting on W, but only the $\mathrm{SL}(4;\mathbb{C})$ subgroup acts on the $N = 4$ theory. There's a nice way of seeing this from the super twistor space point of view: only $\mathrm{SL}(4)$ preserves the trivialization of the Berezinian which we had to fix in order to define the shifted symplectic structure on the algebraic Chern-Simons theory moduli space.

In order to define a *twist* of a theory with an action of this supersymmetry algebra, we identify a supercharge Q (i.e. a fermionic element of the algebra) such that $Q^2 = 0$. Equivalently Q should span a one-dimensional fermionic subalgebra of $\mathcal{A}^{N=4}$. Let's discuss the supercharges we'll use.

Definition 1.20. A *holomorphic* supercharge is a square-zero Q such that the image of $[Q, -]$ is half-dimensional inside of $\mathbb{C}^4$. We can easily find a holomorphic supercharge by taking a rank one element of the summand $S_+ \otimes W$, i.e. an element of the form $Q_{\mathrm{hol}} = \alpha \otimes w$ where $\alpha \in S_+$ and $w \in W$. All such elements square to zero and are holomorphic.

Definition 1.21. A *topological* supercharge is a square-zero Q such that the image of $[Q, -]$ is all of $\mathbb{C}^4$.

Let's analyse the topological supercharges that arise as "further twists" of a holomorphic supercharge Q_{hol}. That is topological supercharges of the form $Q_{\mathrm{hol}} + Q'$ where Q' commutes with Q_{hol} but is not the image of Q_{hol} under ome bosonic symmetry. Equivalently these are topological supercharges coming from the $[Q_{\mathrm{hol}}, -]$-cohomology of $\mathcal{A}^{N=4}$.

It turns out there is, up to R-symmetry, a $\mathbb{CP}^1$-family of such topological further twists. To write it explicitly, choose bases $\langle \alpha_1, \alpha_2 \rangle$, $\langle \beta_1, \beta_2 \rangle$ and $\langle, w_1, w_2, w_3, w_4 \rangle$ for S_+, S_- and W respectively. Let $Q_{\mathrm{hol}} = \alpha_1 \otimes w_1$. Then the family of further topological twists is given by

$$Q_{\mathrm{hol}} + \lambda \alpha_2 \otimes w_2 + \mu(\beta_1 \otimes w_3^* - \beta_2 \otimes w_4^*)$$

where $(\lambda : \mu)$ is a point in $\mathbb{CP}^1$.

Remark 1.22. We should say something briefly about the Kapustin-Witten twisting homomorphism: the block diagonal embedding $\phi_{\mathrm{KW}} \colon \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{sl}(4;\mathbb{C})$. We can use this to modify the $\mathrm{SO}(4)$ action on the $N = 4$ theory in order to define the twisted theories on more general curved manifolds (the untwisted theory is only defined along with its supersymmetry action on framed manifolds). We should note however that the holomorphic supercharge, and in fact all but the point $Q_B = \alpha_1 \otimes w_1 - \alpha_2 \otimes w_2$, are now invariant for the full twisted SO(4)-action, but only for U(2), or for SO(2) × SO(2). So they can be defined on arbitrary complex manifolds but not in full generality.

1.4 Global Twists of Classical Field Theories

Before we talk about our key examples, let's talk about what it means to *twist* a derived stack (building on the better understood notion of a perturbative twist, which I talked about yesterday but which I'll quickly review today). Suppose we're given an action of the group $\mathbb{C}^\times \ltimes \mathbb{C}[-1]$, i.e. an odd symmetry Q such that $Q^2 = 0$, and an action α of $\mathbb{C}^\times$ such that Q has weight 1. In supersymmetric field theories as we've discussed in the example of 4d $N = 4$, one has an action of the group of supersymmetries, and one can often find many copies of $\mathbb{C}^\times \ltimes \mathbb{C}[-1]$ embedded inside it, as we just discussed. If you take the tangent complex $\mathbb{T}_p\mathcal{M}$ at any point in the moduli space of classical solutions, the action of this supergroup is the same as a dg-structure, i.e. a grading (the α-weight) and a differential (Q) of degree 1.

Definition 1.23. To *twist* this tangent complex means to take the total complex for its internal grading and differential and this new grading and differential.

Remark 1.24. We can talk about these perturbative twists from another point of view. One can think of a supersymmetric theory with a chosen square-zero supercharge Q as a family of theories living over the classifying space $B(\Pi\mathbb{C})$, or equivalently a theory with an action of the ring $\mathbb{C}[[t]]$ where t is a fermionic parameter of degree 1. The $\mathbb{C}^\times$ action allows us to define a twisted theory by inverting the parameter t – or equivalently restricting to the formal punctured disk – then taking $\mathbb{C}^\times$-invariants. This is equivalent to the recipe I just described (see Costello [Cos13] where this construction was first given in this kind of language).

One can define twists of the whole derived stack $\mathcal{M}$ in the following setting. Fix a base derived stack $\mathcal{B}$, and nilisomorphisms $\pi\colon \mathcal{M} \leftrightarrows \mathcal{B}\colon \sigma$ (that is, maps that induce isomorphisms after taking H^0), where σ is a section of π. That is, $\mathcal{M}$ is a *pointed formal moduli problem* over $\mathcal{B}$. In our examples coming from gauge theory, $\mathcal{B} = \mathrm{Bun}_G(X)$ – we call such theories *formal algebraic gauge theories*. One can then show the following.

Proposition 1.25. If $\pi\colon \mathcal{M} \leftrightarrows \mathcal{B}\colon \sigma$ is a pointed formal moduli problem over $\mathcal{B}$, and $\mathcal{M}$ admits an action (α, Q) of $\mathbb{C}^\times \ltimes \mathbb{C}[-1]$ so that σ is equivariant for the trivial action on $\mathcal{B}$, then there exists a formal moduli problem $\mathcal{M}^Q \leftarrow \mathcal{B}\colon \sigma^Q$ under $\mathcal{B}$ whose relative tangent complex at each point $b \in \mathcal{B}$ is obtained by twisting the relative tangent complex of σ.

Remark 1.26. How does this work? Well, Gaitsgory and Rozenblyum [GR] prove that pointed formal moduli problems are uniquely determined by their relative tangent complex, as a sheaf of (dg) Lie algebras, and unpointed formal moduli problems (where there is only a map σ, not π) are determined by the tangent Lie algebroid. One can use the perturbative definition above to twist the relative tangent complex, possibly breaking the map π, to obtain a new Lie algebroid, then use Gaitsgory and Rozenblyum's equivalence to obtain a new formal moduli problem.

Example 1.27. If S is a smooth scheme, the derived stack $T[1]S$ has a canonical family of twists, where Q acts as the differential $\lambda \times \mathrm{id}\colon \mathbb{T}_sS[1] \to \mathbb{T}_sS$ on the tangent complex. The twist with respect to this Q is denoted $S_{\lambda\text{-dR}}$: when $\lambda = 1$ this is the *de Rham stack* of S.

Our starting point is the following *holomorphically twisted* $N = 4$ theory. This is the minimal twist of $N = 4$ super Yang-Mills theory that can be defined algebraically.

Theorem 1.28. The *holomorphically twisted $N = 4$ Yang-Mills theory* on a smooth proper complex surface X is the derived stack $T^*[-1]\mathrm{Higgs}_G(X)$, with its canonical shifted symplectic structure. Here, $\mathrm{Higgs}_G(X) = \underline{\mathrm{Map}}(T[1]X, BG)$ is the stack of Higgs bundles on X.

Proof Outline. Let's just give a rough idea of how this is proven. We compute the twist at the level of twistor space – the twisted theory turns out to be localized on a section of the twistor fibration which makes it easy to compute the pushforward to $\mathbb{C}^2$, and to define the theory on more general complex surfaces. All we need to do is to view holomorphic Chern-Simons on twistor space as a sheaf of super dg Lie algebras over $\mathrm{Bun}_G(\mathbb{CP}^3)$, and to compute the cohomology of this sheaf with respect to Q_{hol} and an appropriate R-symmetry circle. The resulting cohomology looks like the complex $\Omega^{\bullet,\bullet}(\mathbb{C}^2, \mathfrak{g})$ – which is the shifted tangent complex to the moduli space of Higgs bundles – plus a shift of its dual. □

Now, observe that this can be written in a more symmetrical way! Since X is smooth and proper it has an orientation of dimension 4, and BG has a 2-symplectic structure given by the Killing form, which means, by the AKSZ construction of Pantev-Toën-Vaquié-Vezzosi [PTVV13] there's a (-2)-shifted symplectic structure on $\mathrm{Higgs}_G(X)$. We can therefore write

$$T^*[-1]\mathrm{Higgs}_G(X) \cong T[1]\underline{\mathrm{Map}}(T[1]X, BG).$$

Now, this has a $\mathbb{P}^1$-family of deformations, by deforming either appearance of the Dolbeault stack $T[1]$. This leads to the main theorem of [EY15].

Theorem 1.29 (E-Yoo)**.** The $\mathbb{P}^1$ of twists of the holomorphically twisted $N = 4$ theory coincide with the $\mathbb{P}^1$ of topological twists constructed by Kapustin and Witten.

What does this mean? Well, one can twist a supersymmetric field theory using any square 0 supercharge. We explicitly computed the action of the supersymmetry algebra on the $N = 4$ theory, and hence on its holomorphic twist, and proved that the $\mathbb{P}^1$ of deformations described above coincides with the action of Kapustin and Witten's family of topological supercharges.

Examples 1.30. The most important two points in this $\mathbb{P}^1$ family are $(1 : 0)$ and $(0 : 1)$, which we call the A- and B-twists respectively. The spaces we obtain in these two twists are $\mathrm{Higgs}_G(X)_{\mathrm{dR}}$ and $T^*[-1]\mathrm{Flat}_G(X)$.

2 Lecture 2 – A Local Description in 2 and 4d, and Quantization

Yesterday I explained how to associate derived moduli stacks to the Kapustin-Witten twists of $N = 4$ gauge theory: they're the moduli spaces of classical solutions to the equations of motion after turning on the appropriate twist. Today I'll start to talk about quantization. First I'll discuss the perturbative quantization (that is, the quantization of the $\mathbb{P}_0$-algebra of local observables). We'll see that this isn't very interesting, and say something about what the quantum theories should assign to an algebraic surface Σ. Along the way we'll talk a little bit about geometric Langlands.

2.1 The A- and B-twisted Theories Locally

So we concluded our discussion yesterday by computing the moduli stacks our Kapustin-Witten twisted theories assign to a complex surface X. Let's continue by discussing what these theories look like in positive codimension.

Let's begin with the B-twist. We write $\mathbb{D} = \mathrm{Spec}\,\mathbb{C}[[t]]$ and $\mathbb{D}^\times = \mathrm{Spec}\,\mathbb{C}((t))$ for the formal disk and the formal punctured disk respectively.

Proposition 2.1. The Kapustin-Witten B-twisted assigns the following derived stacks to spaces in positive codimension:

$$\begin{aligned}
\mathrm{EOM}_B(\mathbb{D}^\times \times \Sigma) &\cong T^*\mathrm{Flat}_G(\Sigma) \\
\mathrm{EOM}_B(\mathbb{D} \times \Sigma) &\cong T^*[1]\mathrm{Flat}_G(\Sigma) \\
\mathrm{EOM}_B(\mathbb{D} \times \mathbb{D}) &\cong T^*[3]BG \cong \mathfrak{g}^*[2]/G.
\end{aligned}$$

To see why this is the case, remember that we compute the twist of an algebraic gauge theory by identifying the untwisted theory with a shef of Lie algebras over $\mathrm{Bun}_G(\Sigma)$. Running this argument starting with the holomorphically twisted $N = 4$ gauge theory, the sheaf of Lie algebras in question splits as a sum of the Lie algebra determining the moduli of Higgs bundles (namely the complex $\Omega^{\bullet,\bullet}(X; \mathfrak{g}_P)$) plus a copy of its dual, shifted by -1. We can identify the dual of the complex of (p, q) forms on each of these spaces with a shift of the same complex, but where the shift depends on the space we choose. We thus obtain, in the twist, a shifted cotangent space where the shift depends on the codimension in which we're working.

Remark 2.2. We can identify the spaces that occur in the A-twist or in a mixed twist in a similar way. In the A-twist things are less interesting because the cotangent disappears after taking the de Rham stack, but it's natural to think of $\mathrm{EOM}_A(\mathbb{D} \times \Sigma)$ as $(\mathrm{Higgs}_G(\Sigma))_{\mathrm{dR}}$, and $\mathrm{EOM}_A(\mathbb{D} \times \mathbb{D})$ as $(BG)_{\mathrm{dR}}$. In what follows we'll focus mainly on the B-twist.

Now, with this calculation in hand, let's talk about the *classical observables* in our twisted gauge theories (this will provide an example of the formalism I discussed on Monday). The classical observables are nothing but functions on the space of solutions to the equations of motion.

Definition 2.3. The *local classical observables* in the B-twisted theory are given by the algebra of functions on the local solutions to the equations of motion. Namely

$$\mathrm{Obs}^{\mathrm{cl}}_B(B^4) = \mathcal{O}(\mathfrak{g}^*[2]/G) \cong \mathcal{O}(\mathfrak{h}^*[2]/W).$$

What sort of structure does this algebra have, and why should we expect this structure from the point of view of field theory? Well, since $\mathfrak{g}^*[2]/G \cong T^*[3]BG$ is 3-shifted symplectic, its algebra of functions has a 3-shifted *Poisson algebra* structure. This is natural from the point of view of classical topological field theory, as we'll explain, but in fact, for degree reasons, this Poisson structure becomes trivial at the level of global functions of the stack.

Definition 2.4. A $\mathbb{P}_n$*-algebra* (in cochain complexes) is a commutative dga A equipped with a bracket $\{,\}\colon A \otimes A \to A[1-n]$ called an $1-n$-shifted Poisson bracket, which is graded anti-symmetric and satisfies a graded Jacobi identity, and is a biderivation for the product.

Remark 2.5. In particular, the algebra of functions on an n-shifted symplectic stack always comes equipped with a $\mathbb{P}_{n+1}$-algebra structure.

These $\mathbb{P}_n$-algebra structures show up naturally in topological theory because of the following theorem. Recall that the algebra of local observables in a topological field theory can naturally be given the structure of an $\mathbb{E}_n$-algebra.

Theorem 2.6 (Poisson additivity [Saf16])**.** The data of a $\mathbb{P}_n$-algebra is equivalent to the data of a $\mathbb{P}_0$-algebra in the category of $\mathbb{E}_n$ algebras. More succintly there is an equivalence of operads $\mathbb{P}_n \cong \mathbb{P}_0 \otimes \mathbb{E}_n$

As such, the local observables in a classical topological field theory should always have a $\mathbb{P}_n$-algebra structure coming from the (-1)-shifted symplectic structure on the moduli space of solutions to the equations of motion and the factorization structure.

Remark 2.7. We get something very similar on the A-side: the algebra of local observables looks like

$$\mathrm{Obs}^{\mathrm{cl}}_A(B^4) = \mathcal{O}((BG_{\mathrm{dR}}) \cong H^\bullet_{\mathrm{dR}}(BG).$$

The de Rham cohomology of BG is indeed generated by the Cartan algebra $\mathfrak{h}$ (identified with the space of invariant polynomials), but not generally concentrated entirely in degree 2, so there is an ungraded isomorphism between the local observables in the A- and B-twisted field theories. The algebra is however concentrated in non-negative even degrees, which means that there cannot be a non-trivial 3-shifted Poisson bracket.

2.2 Quantization of the Observables

So, we've found that in the B-twisted theory the classical local observables are given by the algebra $\mathcal{O}(\mathfrak{h}^*[2]/W)$ equipped with the trivial $\mathbb{P}_4$-structure. Let's now talk about what it means to *quantize* this algebra. In this local, topological context, the problem of quantization has been studied in detail, so we'll be able to say something very precise.

We gave a concrete definition of a $\mathbb{P}_n$-algebra above, but if n is at least 2 one can instead characterize the $\mathbb{P}_n$-operad as the cohomology of the $\mathbb{E}_n$-operad. In particular, given an $\mathbb{E}_n$-algebra A one can produce an $\mathbb{P}_n$-algebra by taking its cohomology. To put it more precisely we can say the following. For an $\mathbb{E}_n$-algebra $\mathcal{A}$, by definition we have a map $\mathrm{Emb}(\coprod_I B^n, B^n) \times \mathcal{A}^I \to \mathcal{A}$. For $I = \{1,2\}$, we have a map $S^{n-1} \to \mathrm{Emb}(B^n \coprod B^n, B^n)$ by considering the first disk fixed at the origin. This gives a map $S^{n-1} \times \mathcal{A}^2 \to \mathcal{A}$. Taking cohomology, we get a map $\mathrm{H}^\bullet(S^{n-1}) \otimes \mathrm{H}^\bullet(\mathcal{A})^{\otimes 2} \to \mathrm{H}^\bullet(\mathcal{A})$. Thinking of the nontrivial class in $\mathrm{H}^{n-1}(S^{n-1})$, we have a map $\mathrm{H}^\bullet(\mathcal{A})^{\otimes 2}[n-1] \to \mathrm{H}^\bullet(\mathcal{A})$, or $(\mathrm{H}^\bullet(\mathcal{A})[n-1])^{\otimes 2} \to \mathrm{H}^\bullet(\mathcal{A})[n-1]$.

Theorem 2.8 (Cohen [Coh76])**.** Let $\mathcal{A}$ be an $\mathbb{E}_n$-algebra. Then the above map on $\mathrm{H}^\bullet(\mathcal{A})$ induces a Lie bracket of degree $1-n$ on $\mathrm{H}^\bullet(\mathcal{A})$. Moreover, if $n > 1$, then $\mathrm{H}^\bullet(\mathcal{A})$ is a $\mathbb{P}_n$-algebra.

In general, to *quantize* a $\mathbb{P}_0$-factorization algebra means the following. There is an operad over the formal disk $\mathbb{D}_\hbar$ called BD_0 (short for Beilinson-Drinfeld), whose fiber over $\hbar = 0$ is equivalent to $\mathbb{P}_0$ and whose generic fiber is equivalent to $\mathbb{E}_0$. A *quantization* of a $\mathbb{P}_0$-factorization algebra is a lift to a BD_0-factorization algebra which recovers the original theory by evaluating at $\hbar = 0$. In the topological case we can put it a lot more simply. To quantize an $\mathbb{P}_n$-algebra A^{cl} will therefore mean to find a lift to an $\mathbb{E}_n$-algebra A^{q} such that $H^\bullet(\mathcal{A}^{\mathrm{q}}) \cong A^{\mathrm{cl}}$ as a $\mathbb{P}_n$-algebra.

Surprisingly, the process of going from an $\mathbb{E}_n$-algebra to a $\mathbb{P}_n$-algebra by taking cohomology doesn't lose any information for $n \geq 2$. This is articulated by the following formality result of the $\mathbb{E}_n$ operad.

Theorem 2.9. [Toë13, Corollary 5.4] For $n \geq 0$, if $X = \operatorname{Spec} A$, then the dg Lie algebra $C^{\mathbb{E}_{n+1}}(X)[n+1]$ is non-canonically quasi-isomorphic to the dg Lie algebra $\mathrm{Pol}(X,n)[n+1]$ of n-shifted polyvector fields. The quasi-isomorphism depends on the choice of a Drinfeld associator.

Here $\mathrm{Pol}(X,n) = \mathcal{O}(T^*[n+1]X)$ is the $\mathbb{P}_{n+2}$-algebra of shifted polyvector fields. Its shift $\mathrm{Pol}(X,n)[n+1]$ is the dg Lie algebra controlling deformations of the $\mathbb{P}_{n+1}$-algebra structure on A. On the other side, $C^{\mathbb{E}_{n+1}}(X)$ is the $\mathbb{E}_{n+1}$-Hochschild cochain complex, which is an $\mathbb{E}_{n+2}$-algebra. Its shift $C^{\mathbb{E}_{n+1}}[n+1]$ has the structure of a dg Lie algebra controlling the deformations of the $\mathbb{E}_{n+1}$-algebra structure on A. So, this formality theorem is saying that the space of $\mathbb{E}_{n+1}$-deformations of an algebra is equivalent to the space of $\mathbb{P}_{n+1}$-deformations.

Now let's go back to our example of the B-twisted theory. In order to understand quantizations of the (trivial) $\mathbb{P}_4$-structure on our local classical observables, or equivalently $\mathbb{E}_4$-deformations of this trivial structure, we have to compute the dg Lie algebra $\mathrm{Pol}(\mathfrak{h}^*[2]/W, 3)[4]$. More precisely, deformations are given by Maurer-Cartan elements of this dg Lie algebra. However, $\mathrm{Pol}(\mathfrak{h}^*[2]/W, 3)[4] = \mathcal{O}(T^*[4]\mathfrak{h}^*[2]/W)[4]$ is concentrated in even degrees, so there can't be any Maurer-Cartan elements (which live in degree 1). We conclude the following.

Proposition 2.10. The only quantization of the $\mathbb{P}_4$ algebra of classical observables in the B-twisted $N = 4$ theory is the *trivial* quantization. That means we can view $\mathcal{O}(\mathfrak{h}^*[2]/W)$ as the algebra of *quantum* observables in the B-twisted theory.

This is a little disappointing: the factorization algebras associated to the Kapustin-Witten theories are very boring. By passing from $\mathfrak{g}^*[2]/G$ of $(BG)_{\mathrm{dR}}$ to its affinization we threw away too much information. In the rest of this lecture we'll describe a less precise ansatz that describes a non-trivial part of the quantum Kapustin-Witten theories, but we'll see tomorrow that this story requires a correction in terms of this boring algebra of quantum observables, so this calculation will be important even for interesting quantum aspects of the theory!

2.3 Geometric Quantization

2.3.1 The Geometric Langlands Correspondence

I won't have time to really do justice to this large subject today. I'll try to give an impression of what sort of objects are studied, and what relationships they're expected to have, and then we'll observe a connection with the moduli spaces we've been discussing yesterday and today.

Fix a reductive complex algebraic group G and a smooth proper complex algebraic curve Σ. The geometric Langlands program consists of several related conjectures about categorified harmonic analysis on the stack $\mathrm{Bun}_G(\Sigma) = \underline{\mathrm{Map}}(\Sigma, BG)$, the moduli stack of algebraic G-bundles on Σ. At its core, the conjecture says that objects in the category $D(\mathrm{Bun}_G(\Sigma))$ of D-modules on $\mathrm{Bun}_G(\Sigma)$ can be decomposed according to a nice basis, into eigenvectors for certain natural operators, and those eigenvectors are parameterised by a moduli space built out of the *Langlands dual* group $G^\vee$. So the geometric Langlands conjecture is a kind of non-abelian, categorical Fourier transform.

Every complex reductive group G has a Langlands dual group $G^\vee$, the unique group whose roots are the coroots of G, whose characters are the cocharacters of G, and vice versa. For instance, GL_n and $\mathrm{SO}(2n)$ are self-dual, SL_n and PGL_n are dual to one another, as are $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(n)$.

Definition 2.11. The *dual* space to $\mathrm{Bun}_G(\Sigma)$, whose points parameterize eigensheaves on $\mathrm{Bun}_G(\Sigma)$, is the derived stack $\mathrm{Flat}_{G^\vee}(\Sigma) = \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, BG)$ is the derived moduli stack of flat $G^\vee$-bundles on Σ. The word "derived" starts to matter here, for instance if $\Sigma = \mathbb{P}^1$ and $G = \mathbb{G}_m$, $\mathrm{Flat}_{\mathbb{G}_m}(\mathbb{P}^1) \cong \mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt} \times B\mathbb{G}_m$: if you don't take the derived stack you lose that first factor.

The following conjecture isn't the oldest form of the geometric Langlands conjecture, but it's the more modern, categorical form that appears in the work of Kapustin and Witten.

Conjecture 2.12 (The "Best Hope")**.** There is an equivalence of categories

$$D(\mathrm{Bun}_G(\Sigma)) \cong \mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma),$$

which intertwines natural symmetries on the two sides.

Remarks 2.13.

1. A mnemonic for thinking about the geometric Langlands conjecture, at least for $G = \mathrm{GL}_n$, is as follows.

The category *of flat bundles on the* space *of vector bundles is equivalent to the* category *of vector bundles on the* space *of flat bundles.*

   This should look familiar when you think about the $\mathbb{CP}^1$ of topological twists of the holomorphic theory: there were two spots where a Dolbeault stack could be deformed to a de Rham stack, so the two dual twists looks like the Dolbeault stack of the moduli of flat bundles, versus the de Rham stack of the moduli of Higgs bundles.

2. If G is abelian, the conjecture is actually a theorem, independently due to Laumon [Lau96] and Rothstein [Rot96]. They realise the equivalence as a twisted version of the Fourier-Mukai transform.

3. Unfortunately, this "best hope" conjecture is false as long as G is non-abelian, one can see this even if $\Sigma = \mathbb{CP}^1$ [Laf12]: the category $\mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma)$ on the B-side is too small (in a precise sense – the geometric Eisenstein series functors fail to preserve compact objects). Arinkin and Gaitsgory [AG12] proposed a corrected form of the geometric Langlands conjecture by enlarging this category in a minimal way. I'll come back to this conjecture in tomorrows lecture.

   Conjecture 2.14 (Arinkin-Gaitsgory)**.** There is an equivalence of categories

   $$D(\mathrm{Bun}_G(\Sigma)) \cong \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Flat}_{G^\vee}(\Sigma),$$

   which intertwines natural symmetries on the two sides.

2.3.2 Categories of Boundary Conditions

Let's connect that geometric Langlands story to our family of topological twists by proposing the following ansatz.

Claim (Categorical Geometric Quantization)**.** Given a 4d classical topological field theory which is a *cotangent theory* – meaning the moduli spaces of solutions to the equations of motion are equivalent to shifted cotangent spaces – a good model for the category of boundary conditions along a surfaces Σ is provided by the category of sheaves on the stack $\mathcal{M}(\Sigma)$, where $\mathrm{EOM}(\Sigma) \cong T^*[1]\mathcal{M}(\Sigma)$ is the shifted cotangent space of solutions to the classical equations of motion.

Remarks 2.15.

1. This ansatz is motivated by the usual story of geometric quantization, where the Hilbert space of a quantum field theory along a manifold of codimension 1 is modelled by the space of sections of the prequantum line bundle which are constant along the leaves of a polarization. In the case where the phase space is actually a cotangent bundle there's a canonical choice of polarization: the Hilbert space is modelled by functions on the base of the cotangent space.
2. There are plenty of subtleties in ordinary geometric quantization which I'm not addressing here. One question which I haven't answered is: what exactly do I mean by the "category of sheaves"? This is a choice that we have to make when describing our model. For reasons we'll discuss in more details tomorrow, in the B-twisted theory I'll consider the category of ind-coherent sheaves, but from the point of view of S-duality it's not clear what the matching choice should be on the A-side. I'll come back to this question.

I'll conclude today by applying this ansatz in the case of the Kapustin-Witten twisted $N = 4$ theories.

- In the case of the B-twist, we have $\mathrm{EOM}_B(\Sigma) \cong T^*[1]\mathrm{Flat}_G(\Sigma)$, which means
$$Z_B(\Sigma) = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma)).$$
- For the A-twist, we have $\mathrm{EOM}_A(\Sigma) \cong (\mathrm{Higgs}_G(\Sigma))_{\mathrm{dR}} \cong T^*[1](\mathrm{Flat}_G(\Sigma))_{\mathrm{dR}}$, so our ansatz gives
$$Z_A(\Sigma) = \mathrm{IndCoh}(\mathrm{Bun}_G(\Sigma)_{\mathrm{dR}}) = \text{D-mod}(\mathrm{Bun}_G(\Sigma)).$$

So we're seeing the two categories appearing in the (best hope version of) the geometric Langlands conjecture! I haven't really talked about S-duality in this lecture series, but Kapustin and Witten's program revolves around the fact that dual twists of $N = 4$ gauge theory, with dual gauge groups, are interchanged by a certain *duality* of quantum field theories. In particular this duality should yield an equivalence of the categories of boundary conditions on a curve Σ, which leads Kapustin and Witten to conclude that the geometric Langlands correspondence arises as a consequence of S-duality.

In tomorrow's lecture I'll explain how the Arinkin-Gaitsgory singular support condition fits into this physical story.

3 Lecture 3 – The Category of Boundary Condition, Vacua and Singular Supports

My plan today is to conclude this lecture series with a discussion of the connection between the singular support conditions of Arinkin and Gaitsgory [AG12] and the Kapustin-Witten twisted gauge theories we've been discussing. I'll argue that the singular support conditions occur by considering support conditions for the action of the local observables that we discussed yesterday on the categories of boundary conditions. This story comes from my joint work [EY17] with Philsang Yoo.

With this in mind I'll start out with some background, on how to think about the action of the local observables on the categories of boundary conditions, on what exactly these singular support conditions are, and then we'll discuss how the two notions are related. I'll conclude today with some conjectures and open questions.

3.1 Local Operators and the Moduli of Vacua

Let's begin today by talking about the action of an $\mathbb{E}_2$-algebra on a category. This will be important because it naturally shows up in the physical story: there's an $\mathbb{E}_2$-action of the algebra of local observables on the category of boundary conditions assigned to a surface in any 4d topological field theory.

Choose any dg-category $\mathcal{C}$. There's automatically an action of the monoidal category $\mathrm{HC}^\bullet(\mathcal{C})$-mod on $\mathcal{C}$, and this action is universal – if you like you can take this universal property as a definition of the Hochschild cochains. For *any* cdga A, an action of the category A-mod on $\mathcal{C}$ is equivalent to a homomorphism $A \to \mathrm{HC}^\bullet(\mathcal{C})$.

Example 3.1. Suppose $\mathcal{C}$ is the category of *boundary conditions* in an n-dimensional topological quantum field theory along a smooth compact manifold of dimension $n-2$ (for instance, in the context of extended functorial TQFT). There's always an action of the algebra A of *local observables* of the field theory on $\mathcal{C}$. In a topological field theory the local observables A naturally form an $\mathbb{E}_n$-algebra, but the cohomology $H^\bullet(A)$ admits an ordinary graded commutative product.

We think about this action as follows. Let $\mathcal{B}$ be the category of boundary conditions along M in a topological field theory, and let $\mathcal{F}$ be an object in $\mathcal{B}$. The algebra $\mathrm{End}_\mathcal{B}(\mathcal{F})$ describes the space of local observables in the *bulk-boundary theory* associated to think boundary condition, and there's a tautological map from bulk observables into bulk-boundary observables for any choice of boundary condition.

On the other hand, and in the other extreme, we should view the Hochschild cochains as the *2d local operators*: that is, the algebra of operators in the 2d theory obtained from compactification along M. Another way of seeing the action of local operators is by observing that any local operator can be viewed as a 2d local operator (if you like, this is part of the factorization structure).

If a commutative algebra A acts on a category $\mathcal{C}$, the *support* of an object in $\mathcal{C}$ is a closed subset of $\mathrm{Spec}\, A$. This has a natural physical meaning in the case where A is the algebra of local observables in a topological field theory. It goes like this.

Definition 3.2. *States* in a quantum field theory on $\mathbb{R}^n$ with algebra $\mathrm{Obs}^q(B^n)$ of local observables are functionals $\phi\colon \mathrm{Obs}^q(B^n) \to \mathbb{R}$. A state ϕ is a *vacuum state* if it translation invariant and satisfies the *cluster decomposition property*, which says $\mathcal{O}_1$ on $B_{r_1}(0)$ and $\mathcal{O}_2$ on $B_{r_2}(0)$, we have

$$(\mathcal{O}_1 * \tau_x(\mathcal{O}_2))(\phi) - \mathcal{O}_1(\phi)\mathcal{O}_2(\phi) \to 0 \text{ as } x \to \infty$$

where τ_x denotes the translation of an observable by $x \in \mathbb{R}^n$. In a *topological* field theory this just says that ϕ is a *ring homomorphism*, so vacuum states are nothing but (closed) points in the spectrum $\mathrm{Spec}(\mathrm{Obs}^q(B^n))$.

So, to summarise what we just explained, in a 4d topological field theory, the category $Z(\Sigma)$ assigned to a surface Σ is acted on by the algebra of local observables, so we can talk about the *support* of an object in the moduli space of vacua. Let's talk about what this support means.

Remark 3.3. We motivate this definition in the following way. Objects in the category $\mathcal{B}_v$ of boundary conditions with support $\{v\}$ are objects $\mathcal{F}$ of $\mathcal{B}$ so that $\mathrm{End}_\mathcal{B}(\mathcal{F})$ is supported at v, just as in the previous subsection. What does this mean from the point of view of vacua? Well, we should think of the space $\mathrm{End}_\mathcal{B}(\mathcal{F})$ as the *phase space* of our topological field theory coupled to the boundary condition $\mathcal{F}$. Indeed, in general, the hom space $\mathrm{Hom}_\mathcal{B}(\mathcal{F}_1, \mathcal{F}_2)$ can be interpreted as the space of states on a strip $Y^{n-2} \times [0,1]$ with boundary conditions $\mathcal{F}_1$ and $\mathcal{F}_2$ on the two boundary components. In particular we obtain $\mathrm{End}_\mathcal{B}(\mathcal{F})$ by putting the boundary condition $\mathcal{F}$ at both boundary components. Alternatively (assuming we're working in a topological context) we can view this as the space of observables on Y times a half disk $D = \{(x,y)\colon x^2+y^2 \leq 1, x > 0\}$, with boundary condition $\mathcal{F}$ on the closed edge (see Figure 1). That these descriptions are equivalent is a version of the state-operator correspondence.

From this point of view there's clearly a map from the algebra A of observables in the bulk to the algebra $\mathrm{End}_\mathcal{B}(\mathcal{F})$ of observables in this coupled bulk-boundary system by the inclusion of those observables supported at a small ball in the interior of $D \times Y$. In particular using this inclusion one can evaluate elements of $\mathrm{End}_\mathcal{B}(\mathcal{F})$ at vacua in $\mathcal{V} = \operatorname{Spec} A$. Thus, from this point of view, an object $\mathcal{F}$ survives the restriction if the bulk-boundary phase space $\operatorname{End}_{\mathcal{B}}(\mathcal{F})$ is supported at v, in other words if the bulk-boundary system is acted on non-trivially by those observables that only depend on a small neighborhood of $v \in \mathcal{V}$. Then objects of the restricted category $\mathcal{B}_v$ are those boundary conditions that can "see" the vacuum state v.

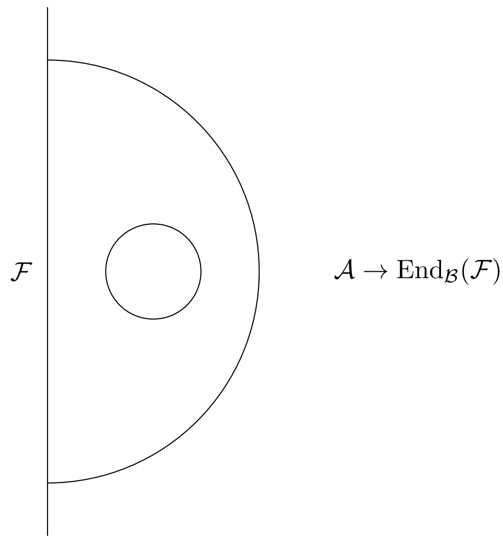

Figure 1: The action of $\mathcal{A}$ on $\mathcal{F}$ is mediated by a whistle cobordism with the given boundary condition.

3.1.1 Vacua in the B-Twisted Theory

Having introduced the notion of vacua in general, let's talk about our specific example: the Kapustin-Witten B-twist. Recall that the algebra of classical observables is equivalent to $\mathcal{O}(\mathfrak{h}^*[2]/W)$ with a trivial shifted Poisson bracket. We argued yesterday that this algebra doesn't receive any quantum corrections, so it's exact at the quantum level. The moduli space of vacua is therefore just the spectrum of this algebra:

$$\mathcal{V}\text{ac} = \mathfrak{h}^*[2]/W,$$

or to put it another way, the affinization of the moduli space of local solutions to the equations of motion. Because classically this derived stack only has one point, there's only one support condition we can consider: we can consider the full subcategory of boundary conditions supported at the point 0.

As we'll see shortly, this condition is actually very interesting, but we can go a little further: it's possible to *regrade* the moduli space of vacua in order to remove this shift by 2, after which we can talk about objects supported at any point in $\mathfrak{h}^*/W$. There are two ways of doing this, an easy way and a hard way. The easy way is to work in the setting of $\mathbb{Z}/2$-graded derived algebraic geometry (instead of $\mathbb{Z}$-graded), in which context the even shift obviously vanishes. Let's say something about the more subtle regrading procedure, although it's a bit of a technical aside so I won't spend too much time on it.

Recall that when we explained what it meant to twist a derived stack – in particular in Remark 1.24 – we could view the untwisted theory with its Q-action as a family of theories over $B(\Pi\mathbb{C})$. We could then define the twist by inverting th parameter t and taking $\mathbb{C}^\times$-invariants. What is this is too restrictive: let's not take those invariants, but instead keep the parameter t in place, though we might restrict to the bosonic part generated by t^2 in order to replace a fermionic degree 1 parameter with a bosonic degree 2 parameter.

If we consider this new theory, the new algebra of local observables looks like $\mathcal{O}(\mathfrak{h}^*[2]/W)((t))$. It doesn't make sense to take Spec of this ring because it's not concentrated in degrees ≤ 0, but we can still talk about its action on the category of boundary conditions, which now has form $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma)) \otimes_{\mathrm{Vect}} \mathbb{C}((t))$-mod, and the support of a boundary condition in Spec of H^0 of the ring of local operators. This is where the regraded scheme appears:

$$\mathrm{Spec}\, H^0(\mathcal{O}(\mathfrak{h}^*[2]/W)((t))) \cong \mathfrak{h}^*/W$$

now with no shift.

3.2 Singular Support Conditions

3.2.1 What Singular Support Means

Now, let's move on to the other ingredient of today's talk: singular support conditions as defined by Arinkin and Gaitsgory. Let's focus on the example where the category $\mathcal{C}$ is the category $\mathrm{IndCoh}(X)$ of ind-coherent sheaves on some space X (more precisely, a derived stack). *Singular supports* are defined using the action of a certain algebra $\mathcal{O}(\mathrm{Sing}(X))$ that maps into the Hochschild cochains.

Definition 3.4. Let X be a derived stack. The *scheme of singularities* of X is the classical part of the -1-shifted cotangent space

$$\mathrm{Sing}(X) = (T^*[-1]X)^{\mathrm{cl}}.$$

Suppose X is a (quasi-smooth) affine derived scheme. Then there's a canonical algebra map (that doesn't respect the grading – there's a degree shift that we'll mention again later) $\mathcal{O}(\mathrm{Sing}(X)) \to \mathrm{HC}^\bullet(X)$.

Definition 3.5. If Y is a closed conical subset of $\mathrm{Sing}(X)$ then the category of sheaves on X with *singular support* in Y is the tensor product

$$\mathrm{IndCoh}_Y(X) = \mathrm{IndCoh}(X) \otimes_{\mathrm{QC}(\mathrm{Sing}(X))} \mathrm{QC}(\mathrm{Sing}(X))_Y.$$

If X is a more general derived stack then we can define the category of sheaves with singular support in $Y \subseteq \mathrm{Sing}(X)$ via smooth descent (take a limit over all smooth maps $Z \to X$ whose source is an affine derived scheme).

Example 3.6. From the point of view of quantum field theory, the category $\mathrm{IndCoh}(X)$ describes a completed version of the category of boundary conditions in the 2d B-model with target X. Singular support conditions will admit a nice physical description from this point of view which we'll explain shortly – one should interpret the algebra $\mathcal{O}(\mathrm{Sing}(X))$ as the algebra of local operators in the 2d B-model (up to one of these ubiquitous degree shifts by two).

So I've nearly explained to you what the category $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is. All I have to do is tell you what the subset $\mathcal{N}_{G^\vee} \subseteq \mathrm{Sing}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is. First I'll describe the space of singularities of $\mathrm{Flat}_{G^\vee}(\Sigma)$. This is a straightforward computation using the fact that the tangent complex to $\mathrm{Flat}_{G^\vee}(\Sigma)$ is the de Rham complex of Σ with coefficients in $(\mathfrak{g}^\vee)^*$, shifted down in cohomological degree by one.

Proposition 3.7. The space $\mathrm{Sing}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is the (classical) moduli stack whose closed points are triples (P, ∇, ϕ) where (P, ∇) is a classical point of $\mathrm{Flat}_{G^\vee}(\Sigma)$ (i.e. a $G^\vee$-bundle with flat connection) modulo gauge transformations, and ϕ is a flat section

$$\phi \in H^0_\nabla(\Sigma; (\mathfrak{g}^\vee)^*_P)$$

of the coadjoint bundle of P. We call this space $\mathrm{Arth}_{G^\vee}(\Sigma)$ – the stack of $G^\vee$-*Arthur parameters* of Σ.

Definition 3.8. The *global nilpotent cone* $\mathcal{N}_{G^\vee} \subseteq \mathrm{Arth}_{G^\vee}(\Sigma)$ is the substack consisting of Arthur parameters (P, ∇, ϕ) where the value ϕ_x of ϕ at a point $x \in \Sigma$ is *nilpotent* as an element of the dual Lie algebra $(\mathfrak{g}^\vee)^*$. This condition doesn't depend on the choice of point x because the section ϕ was a flat section.

So now we know what the Arinkin-Gaitsgory category $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$ means. It's the category of ind-coherent sheaves whose singular support lies in the global nilpotent cone. My goal for the remainder of this talk will be to explain how this condition arises from quantum field theory.

3.2.2 Singular Support as a Vacuum Condition

Naïvely, we describe the category of boundary conditions in the B-twisted theory by $\mathrm{Coh}(\mathrm{Flat}_G(\Sigma))$, or rather by its completion $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$, as in the usual description of the B-model. The action of local observables on this category has a nice description in terms of geometric representation theory. If we choose a point $x \in \Sigma$ the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ becomes a module for the category of *line operators*, which is given by

$$\mathcal{L} = \mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))$$

where $\mathbb{B} = \mathbb{D} \cup_{\mathbb{D}^\times} \mathbb{D}$ is the "formal bubble" obtained by gluing two formal disks together along a formal punctured disk. This monoidal category acts by convolution – double a formal neighbourhood of the point $x \in \Sigma$ and pull-tensor-push along the diagram

$$\begin{array}{ccccc} & & \mathrm{Flat}_G(\mathbb{B}) & & \\ & & \uparrow{\scriptstyle q_x} & & \\ & & \mathrm{Flat}_G(\Sigma \amalg_{\mathbb{D}^\times} \mathbb{D}_x) & & \\ & \swarrow{\scriptstyle p_1} & & \searrow{\scriptstyle p_2} & \\ \mathrm{Flat}_G(\Sigma) & & & & \mathrm{Flat}_G(\Sigma). \end{array}$$

In geometric representation theory the category $\mathcal{L}$ is called the "spectral Hecke category". The monoidal unit of $\mathcal{L}$ is given by the skyscraper sheaf at the trivial bundle. If one computes its endomorphism algebra in $\mathcal{L}$ one sees our algebra A of local operators:

$$\mathrm{End}_{\mathcal{L}}(\delta_1) \cong \mathcal{O}(\mathfrak{h}^*[2]/W) = A.$$

This isn't so surprising – morphisms between two line operators should be given by states on a strip compatible with these line operators on two sides, and if the line operator on both sides this just gives all states, or all local operators under the state-operator correspondence.

The upshot to all this is that we obtain our action of the algebra A of local operators as follows.

$$\begin{aligned} &\text{The action defines a functor } \mathcal{L} \to \mathrm{End}(\mathcal{B}) \\ &\qquad \text{which induces a map } A = \mathrm{End}_{\mathcal{L}}(\delta_1) \to \mathrm{End}_{\mathrm{End}_{\mathcal{B}}}(\mathrm{id}_{\mathcal{B}}) = \mathrm{HC}^\bullet(\mathcal{B}) \\ &\qquad \text{and therefore a map } A \to \mathrm{End}_{\mathcal{B}}(\mathcal{F}) \end{aligned}$$

for each object $\mathcal{F}$ by the universal property of Hochschild cochains. The second line came from the first line by applying the functor to the algebra of endomorphisms of the unit on each side.

We can now state the main result.

Theorem 3.9 (E-Yoo)**.** The full category of boundary conditions compatible with the vacuum $0 \in \mathfrak{h}^*[2]/W$ is equivalent to Arinkin and Gaitsgory's nilpotent singular support category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$.

There's a simple reason that leads us to expect such a result. There's a natural map – evaluation at a point $x \in \Sigma$ from $\mathrm{Arth}_G(\Sigma)$ to $\mathfrak{g}^*/G$. Post-composing with the eigenvalue map defines a map $\mathrm{Arth}_G(\Sigma) \to \mathfrak{h}^*/W$. The action of $\mathcal{O}(\mathfrak{h}^*[2]/W)$ on the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ factors through the natural action of $\mathcal{O}(\mathrm{Arth}_G(\Sigma))$ by which we define singular support, by pullback along the eigenvalue map (note that this is only a map of *ungraded* commutative rings, one needs to be more careful to keep track of all the shifts by two). What's more, the global nilpotent cone can by thought of as coming from the following pullback:

$$\begin{array}{ccc} \mathcal{N}_G & \longrightarrow & \mathrm{Arth}_G(\Sigma) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_x} \\ \{0\} & \longrightarrow & \mathfrak{h}^*/W, \end{array}$$

which means being supported at 0 in $\mathfrak{h}^*/W$ is equivalent to being supported on $\mathcal{N}_G$ in $\mathrm{Arth}_G(\Sigma)$.

3.3 Some Questions and Conjectures

By replacing $\mathcal{O}(\mathfrak{h}^*[2]/W)$ by its shifted version $\mathcal{O}(\mathfrak{h}^*/W)$ it makes sense to ask for the category of boundary conditions compatible with *any* vacuum $v \in \mathfrak{h}^*/W$. We can compute this category by a similar method to the one I just described, and we conjecture that the results fit together in a nice way: that the categories one obtains are equivalent to Arinkin-Gaitsgory categories with the symmetry group broken to a subgroup compatibly with the vacuum. We conjecture the following (and we have some evidence supporting the conjecture).

Conjecture 3.10 (Gauge symmetry breaking)**.** The full subcategory of objects in $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ compatible with the vacuum $v \in \mathfrak{h}^*/W$ is equivalent to $\mathrm{IndCoh}_{\mathcal{N}_L}(\mathrm{Flat}_L(\Sigma))$, where $L \subseteq G$ is the stabilizer of v in $\mathfrak{g}^*$.

Remark 3.11. We can prove that the boundary conditions compatible with $v \in \mathfrak{h}^*/W$ are described by a singular support condition, namely that the singular support lies in the pullback $\mathrm{Arth}_G^v(\Sigma) = \mathrm{Arth}_G(\Sigma) \times_{\mathfrak{h}^*/W} \{v\}$, defined by the map

$$\mathrm{Arth}_G(\Sigma) \overset{\mathrm{ev}_x}{\to} \mathfrak{g}^*/G \to \mathfrak{h}^*/W$$

where the first map evaluates the Arthur parameter at a point $x \in \Sigma$ and the second map is the usual characteristic polynomial map. The content of the above conjecture is the claim that this singular support condition is equivalent to the $\mathcal{N}_L$ condition I described. We show that in fact on the level of geometry, not only is $\mathrm{Arth}_G^v(\Sigma)$ equivalent to $\mathcal{N}_L$, but their formal neighbourhoods in $\mathrm{Arth}_G(\Sigma)$ and $\mathrm{Arth}_L(\Sigma)$ respectively coincide.

Remark 3.12. This conjecture leads us to conjecture something stronger, that there's a *factorization structure* on the Arinkin-Gaitsgory categories. More concretely we make a conjecture about their Hochschild cohomology. We conjecture that there's a factorization algebra whose fiber over $x \in \mathrm{Ran}(x)$ is the direct sum of the algebras

$$\bigoplus_{n \geq 1} \bigoplus_{\widetilde{x} \in \mathrm{Sym}^n(\mathbb{C})} \mathrm{HC}^\bullet(\mathrm{IndCoh}_{\mathcal{N}_{L_{\widetilde{x}}}}(\mathrm{Flat}_{L_{\widetilde{x}}}))$$

where the sum is over $\widetilde{x}$ which are *lifts* of the point x. This factorization structure has a string theoretic origin, coming from the motion of D3 branes in type IIB string theory. There's an analogous story for D0 branes in a certain twist of type IIA on $\mathbb{R}^2 \times X \times \mathbb{C}$ where X is a Calabi-Yau 3-fold: the motion of D0 branes there appears to produce the factorization structure on cohomological Hall algebras which was constructed by Kontsevich and Soibelman [KS11].

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